CLASSIFICATION OF LOG SMOOTH TORIC DEL PEZZO PAIRS

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Abstract. We give a description of all log-Fano pairs (X, D) where X is a smooth toric surface and D a reduced simple normal crossing divisor such that D is a torus invariant divisor.

1. Introduction

The Enriques–Kodaira classification gives a classification of complex compact surfaces using their Kodaira dimension. Nonsingular projective minimal surfaces with Kodaira dimension $-\infty$ have an important position in this problem of classification, they correspond in the MMP-terminology to *Mori's fiber spaces* [5, Theorem 1.5.5]. In this paper, we are interested by log smooth toric del Pezzo pairs.

A n-dimensional *toric variety* is an irreducible variety X containing a torus $T \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T on itself extends to an algebraic action of T on X. Given a simple normal crossing divisor D on X, we say that (X, D) is a *log smooth toric del Pezzo pair* if X is a smooth toric surface and D a torus-invariant divisor such that $-(K_X + D)$ is ample.

Maeda [4] gives a classification of logarithmic Del Pezzo surfaces using Fujita's classification theorem of polarized varieties of Δ-genera zero [2]. In this paper, we give a proof of this classification on toric surfaces using their fans or polytopes. We denote by $\mathbb{F}_r$ the Hirzebruch suface $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(r))$.

Theorem 1.1. *Let X be a smooth complete toric surface and D a reduced torus-invariant divisor on X. Then, the pair (X, D) is log Del Pezzo if:*

1. *$X = \mathbb{P}^2$ and $D = D'$ where D' is a line;*
2. *$X = \mathbb{P}^2$ and $D = D' + D''$ where D' and D'' are two lines;*
3. *$X = \mathbb{F}_r$ and $D = D'$ where D' is a section with $(D')^2 = -r$;*
4. *$X = \mathbb{F}_r$ and $D = D' + D''$ where D' is a section with $(D')^2 = -r$ and D'' is a fiber;*
5. *$X = \mathbb{F}_1$ and $D = D'$ where D' is a section such that $(D')^2 = 1$;*
6. *$X = \mathbb{F}_0$ and $D = D''$ where D'' is a fiber.*

The paper is organized as follows: in Section 2, we gives some properties of polarized toric surfaces and their polytopes and in Section 3, we give the proof of Theorem 1.1.

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2. Ample divisors on toric surfaces

2.1. **Toric varieties.** Let N be a rank n lattice and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual with pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$. Then N is the *lattice of one-parameter subgroups* of the n-dimensional complex torus $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$. We call M the *lattice of characters* of T_N. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we define $N_{\mathbb{K}} = N \otimes_{\mathbb{Z}} \mathbb{K}$ and $M_{\mathbb{K}} = M \otimes_{\mathbb{Z}} \mathbb{K}$. A *fan* Σ in $N_{\mathbb{R}}$ is a set of rational strongly convex polyhedral cones in $N_{\mathbb{R}}$ such that:

- Each face of a cone in Σ is also a cone in Σ;
- The intersection of two cones in Σ is a face of each.

A cone σ in $N_{\mathbb{R}}$ is *smooth* if its minimal generators form part of a $\mathbb{Z}$-basis of N. A fan Σ is *smooth* if every cone σ in Σ is smooth. The *support* of Σ is given by $|\Sigma| := \bigcup_{\sigma\in\Sigma}\sigma$ and we say that Σ is *complete* if $|\Sigma| = N_{\mathbb{R}}$.

Notation 2.1. For a finite subset $S \subseteq N_{\mathbb{R}}$, we denote by $\mathrm{Cone}(S)$ the cone generated by S. For a fan Σ, we denote by $\Sigma(r)$ the set of r-dimensional cones of Σ and by $u_\rho \in N$ the minimal generator of the ray $\rho \in \Sigma(1)$.

Let X be the toric variety associated to a fan Σ in $N_{\mathbb{R}}$ with torus $T = N \otimes_{\mathbb{Z}} \mathbb{C}^*$ [1, Chapter 3]. The variety X is obtained by gluing affine charts $(U_\sigma)_{\sigma\in\Sigma}$ where $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ with $\mathbb{C}[S_\sigma]$ the semi-group algebra of

$$S_\sigma = \sigma^\vee \cap M = \{m \in M : \langle m, u\rangle \geq 0 \text{ for all } u \in \sigma\}.$$

We denote by $O(\sigma)$ the torus-orbit of X associated to $\sigma \in \Sigma$. By the Orbit-cone-correspondence [1, Theorem 3.2.6], there is a bijective correspondence between cones of Σ and torus-orbits of X. Moreover, for any $\sigma \in \Sigma$, $\dim O(\sigma) = \dim(X) - \dim(\sigma)$. Therefore, for any ray $\rho \in \Sigma(1)$, there is a Weil divisor D_ρ defined as the Zariski closure of the orbit $O(\rho)$. As divisors of the form $\sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ are precisely the invariant divisors under the torus action on X, we deduce that

$$\mathrm{WDiv}_T(X) := \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}D_\rho$$

is the group of invariant Weil divisors on X. In particular,

Theorem 2.2 ([1, Theorem 8.2.3])**.** *The canonical divisor of a toric variety X is the torus invariant Weil divisor*

$$K_X = -\sum_{\rho\in\Sigma(1)} D_\rho.$$

We say that X is *smooth* (resp. *complete*) if and only if Σ is smooth (resp. complete). If X is complete, according to [1, Theorem 4.1.3], we have

$$|\Sigma(1)| = \dim(X_\Sigma) + \mathrm{rk}\,\mathrm{Cl}(X_\Sigma)\ . \tag{1}$$

2.2. **Complete toric surfaces.** We assume that $N = M = \mathbb{Z}^2$ and the pairing $\langle\cdot,\cdot\rangle : M \times N \to \mathbb{Z}$ is given by

$$\langle m, u\rangle = a_1b_1 + a_2b_2$$

for $m = (a_1, a_2) \in M$ and $u = (b_1, b_2) \in N$. We denote by (e_1, e_2) be the standard basis of $\mathbb{Z}^2$. A vector $u \in N$ is *primitive* if for all $k > 1$, $\frac{1}{k}u \notin N$. Let Σ be a smooth complete fan in $\mathbb{R}^2$ and X the toric surface associated to Σ. There is a family of primitive vectors $\{u_i \in N : 0 \leq i \leq n-1\}$ with $n \geq 3$ such that

- $\Sigma = \{0\} \cup \{\mathrm{Cone}(u_i) : 0 \leq i \leq n-1\} \cup \{\mathrm{Cone}(u_i, u_{i+1}) : 0 \leq i \leq n-1\}$
- $\det(u_i, u_{i+1}) = 1$

where $u_n = u_0$. For any $i \in \{0, \ldots, n-1\}$, we denote by D_i the divisor corresponding to the ray $\mathrm{Cone}(u_i)$ and we set $\gamma_i = \det(u_{i-1}, u_{i+1})$. By [1, Proposition 6.4.4],

$$\begin{cases} D_i \cdot D_i = -\gamma_i & \\ D_k \cdot D_i = 1 & \text{if } k \in \{i-1, i+1\} \\ D_k \cdot D_i = 0 & \text{if } k \notin \{i-1, i, i+1\} \end{cases}. \tag{2}$$

Let $L = \sum_i a_i D_i$ be a Cartier divisor on X. By the toric Kleiman Criterion (cf. [1, Theorem 6.3.13]), L is ample if and only if for any $i \in \{0, \ldots, n-1\}$,

$$L \cdot D_i = a_{i+1} + a_{i-1} - \gamma_i a_i > 0. \tag{3}$$

The polytope corresponding to L is given by

$$P = \{m \in \mathbb{Z}^2 : \langle m, u_i\rangle \geq -a_i \text{ for } i \in \{0,\dots,n-1\}\} \tag{4}$$

and the facet of P with inward-pointing normal u_i is given by

$$P_i = \{m \in \mathbb{Z}^2 : \langle m, u_i\rangle = -a_i\} \cap P\ . \tag{5}$$

We recall that a lattice M defines a measure ν on $\mathrm{M}_{\mathbb{R}}$ as the pull-back of the Haar measure on $\mathrm{M}_{\mathbb{R}}/\mathrm{M}$. The measure ν is translation invariant and satisfies $\nu(\mathrm{M}_{\mathbb{R}}/\mathrm{M}) = 1$. Let $\mathrm{vol}(P_i)$ be the volume of P_i with respect to the measure determined by $M \cap \{m \in \mathbb{Z}^2 : \langle m, u_i\rangle = -a_i\}$ in its affine span.

Proposition 2.3. *If for all $i \in \{0,\dots,n-1\}$, $P_i \neq \varnothing$, then*

$$\mathrm{vol}(P_i) = |a_{i+1} + a_{i-1} - \gamma_i\, a_i|.$$

Proof. Let $m_i \in M$ such that $\langle m_i, u_i\rangle = -a_i$ and $\langle m_i, u_{i+1}\rangle = -a_{i+1}$. By (5), P_i is the edge having m_{i-1} and m_i for extremities. Therefore, $\mathrm{vol}(P_i) = \mathrm{card}(P_i \cap \mathbb{Z}^2) - 1$.

We first show that, for any $y = (y_1, y_2) \in \mathbb{Z}^2$,

$$\mathrm{card}\left(\{ty : t \in [0;1]\} \cap \mathbb{Z}^2\right) - 1 = \gcd(|y_1|, |y_2|).$$

Let $A = \{ty : t \in [0;1]\} \cap \mathbb{Z}^2$. If $y_1 = 0$, then $\mathrm{card}(A) = y_2 + 1$ and when $y_2 = 0$, $\mathrm{card}(A) = y_1 + 1$. For the case $y_1 \neq 0$ and $y_2 \neq 0$, we can reduce the study to the case where $y_1 > 0$ and $y_2 > 0$. If $\gcd(y_1, y_2) = \ell$, then for any $t \in [0;1]$, $ty \in A$ if and only if $t \in \{k/\ell : k \in \{0, 1, \dots, \ell\}\}$. Therefore, $\mathrm{card}(A) = \ell + 1$.

We write $u_i = \alpha_i\, e_1 + \beta_i\, e_2$ with α_i, $\beta_i \in \mathbb{Z}$. The equations $\langle m_i, u_i\rangle = -a_i$ and $\langle m_i, u_{i+1}\rangle = -a_{i+1}$ give

$$m_i = \begin{pmatrix} a_{i+1}\, \beta_i - a_i\, \beta_{i+1} \\ -a_{i+1}\, \alpha_i + a_i\, \alpha_{i+1} \end{pmatrix}.$$

Hence,

$$\overrightarrow{m_{i-1}\, m_i} = \begin{pmatrix} \beta_i(a_{i+1} + a_{i-1}) - a_i(\beta_{i+1} + \beta_{i-1}) \\ -\alpha_i(a_{i+1} + a_{i-1}) + a_i(\alpha_{i-1} + \alpha_{i+1}) \end{pmatrix}.$$

As $u_{i-1} - \gamma_i\, u_i + u_{i+1} = 0$, we get

$$\overrightarrow{m_{i-1}\, m_i} = \begin{pmatrix} \beta_i(a_{i+1} + a_{i-1}) - a_i\, \gamma_i\, \beta_i \\ -\alpha_i(a_{i+1} + a_{i-1}) + a_i\, \gamma_i\, \alpha_i \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \mathrm{vol}(P_i) &= \gcd(\, |\beta_i(a_{i+1} + a_{i-1} - a_i\, \gamma_i)|, |\alpha_i(a_{i+1} + a_{i-1} - a_i\, \gamma_i)|\,) \\ &= |a_{i+1} + a_{i-1} - a_i\, \gamma_i| \gcd(|\beta_i|, |\alpha_i|) \end{aligned}$$

As u_i is a primitive vector, we get $\gcd(\alpha_i, \beta_i) = 1$ and the desired formula. ☐

Remark 2.4. If P is the polytope corresponding to the polarized toric surface (X, L), then $L \cdot D_i = \mathrm{vol}(P_i)$.

3. Smooth toric log del Pezzo pairs

We use the notations of the previous section. We describe here all log smooth toric del Pezzo pairs. Let X be a toric surface associated to a fan Σ. By Equation (1), we have $\mathrm{card}(\Sigma(1)) = 2 + \mathrm{rk}(\mathrm{Pic}(X))$.

Lemma 3.1. *Let X be a complete smooth toric surface with Picard rank p and D a reduced invariant divisor of X defined by $D = \sum_{i \in \Delta} D_i$ where $\Delta \subseteq \{0,\dots,n-1\}$.*

1. *If $\mathrm{card}(\Delta) \geq 3$, then $-(K_X + D)$ is not ample.*
2. *If $p \geq 3$ and $\mathrm{card}(\Delta) \in \{1, 2\}$, then $-(K_X + D)$ is not ample.*

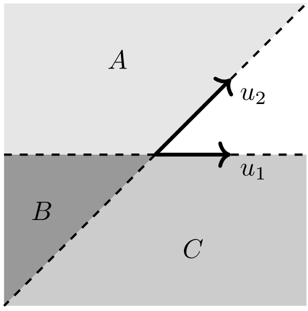

FIGURE 1. Geometry of the fan

Proof. Let $\Delta' = \{0, \ldots, n-1\} \setminus \Delta$. By Theorem 2.2, we have

$$-(K_X + D) = \sum_{i \in \Delta'} D_i.$$

First point. Let P be the polytope corresponding to $-(K_X + D)$. By (5), $0 \in P_i$ for all $i \in \Delta$. Therefore, there is $i \in \{0, \ldots, n-1\}$ such that $\mathrm{vol}(P_i) = 0$. Hence, $-(K_X + D)$ is not ample.

Second point. For the proof of this point, we will use the geometry of the fan. Let $A = \{-\alpha\, u_1 + \beta\, u_2 : \alpha, \beta \geq 0\}$, $B = \{-\alpha\, u_1 - \beta\, u_2 : \alpha, \beta \geq 0\}$ and $C = \{\alpha\, u_1 - \beta\, u_2 : \alpha, \beta \geq 0\}$ pictured in Figure 1.

We start with the case $\mathrm{card}\, \Delta = 1$. We assume that $D = D_1$. We have $-(K_X + D) \cdot D_0 = 1 - \gamma_0$ and $-(K_X + D) \cdot D_2 = 1 - \gamma_2$. If $-(K_X + D)$ is ample, then $\gamma_0 \leq 0$ and $\gamma_2 \leq 0$. As $\gamma_2 = \det(u_1, u_3)$ and $\gamma_0 = \det(u_{n-1}, u_1)$, we deduce that $u_3 \in B$ and $u_{n-1} \in A$. When $n \geq 5$, this is in contradiction with the fact that if $u_3 \in B$, then u_{n-1} is in B or C. Thus, we deduce that $-(K_X + D)$ is not ample.

We now assume that $\mathrm{card}(\Delta) = 2$. After renumbering the indices, we can assume that $D = D_1 + D_j$ with $j \in \{2, \ldots, n-1\}$. We first assume that $j \in \{3, \ldots, n-1\}$. Let P be the polytope of $-(K_X + D)$. As $0 \in P_1$ and $0 \in P_j$, we deduce that 0 is a vertex of P. Hence, for any $k \in \{2, \ldots, j-1\}$, $\mathrm{vol}(P_k) = 0$. By (3), we deduce that $-(K_X + D)$ is not ample.

We now assume that $D = D_1 + D_2$. We have $-(K_X + D) \cdot D_3 = 1 - \gamma_3$ and $-(K_X + D) \cdot D_0 = 1 - \gamma_0$. If $-(K_X + D)$ is ample, then $\gamma_3 \leq 0$ and $\gamma_0 \leq 0$. As $\gamma_3 = \det(u_2, u_4)$ and $\gamma_0 = \det(u_{n-1}, u_1)$, we deduce that $u_4 \in C$ and $u_{n-1} \in A$. If $n \geq 6$, this situation contradicts the positioning order of vectors u_i. If $n = 5$, we have $u_4 \in A$ and $u_4 \in C$, this is not possible. Therefore, we deduce that $-(K_X + D)$ is not ample. □

If $\Delta \neq \varnothing$, according to Lemma 3.1, it is enough to study the positivity of $-(K_X + D)$ when $\mathrm{rk}\, \mathrm{Pic}(X) \in \{1, 2\}$ and $\mathrm{card}(\Delta) \in \{1, 2\}$. Note that the only smooth projective toric surface with Picard number one is the projective space $\mathbb{P}^2$. The rays of the fan of $\mathbb{P}^2$ are the half-lines generated by $u_1 = e_1$, $u_2 = e_2$ and $u_0 = -(e_1 + e_2)$.

Proposition 3.2. *If $X = \mathbb{P}^2$, then the log smooth pair (X, D) is toric log del Pezzo if and only if $D \in \{D_0, D_1, D_2\} \cup \{D_0 + D_1, D_0 + D_2, D_1 + D_2\}$.*

Proof. We have the linear equivalence $D_1 \sim_{\mathrm{lin}} D_0$ and $D_2 \sim_{\mathrm{lin}} D_0$. By Theorem 2.2, we have $K_X = -(D_0 + D_1 + D_2)$, i.e $-K_X \sim_{\mathrm{lin}} 3D_0$. As D_0 is ample, we deduce that $-(K_X + D)$ is not ample if and only if $D = D_1 + D_2 + D_3$. □

By [3, Theorem 1], every smooth toric surfaces of Picard rank two if of the $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(r))$ with $r \in \mathbb{N}$. Let $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(r))$. The rays of the fan of X

are the half lines generated by the vectors $u_1 = e_1$, $u_2 = e_2$, $u_3 = -e_1 + r\,e_2$ and $u_0 = -e_2$. The numbers γ_i are given by $\gamma_0 = -r$, $\gamma_1 = 0$, $\gamma_2 = r$ and $\gamma_3 = 0$. By (3), the divisor $L = a_0\,D_0 + a_1\,D_1 + a_2\,D_2 + a_3\,D_3$ is ample if and only if

$$a_0 + a_2 > 0\ ,\ \ a_1 + a_3 > r\,a_2\ ,\ \ a_1 + a_3 > -r\,a_0$$

if and only if

$$a_0 + a_2 > 0 \quad \text{and} \quad a_1 + a_3 > r\,a_2\ . \tag{6}$$

Proposition 3.3. *Let $X = \mathbb{P}\,(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(r))$ with $r \in \mathbb{N}$. Then :*

(1) $-K_X$ or $-(K_X + D_0)$ are ample if and only if $r \in \{0,\,1\}$.
(2) If $D \in \{D_1,\,D_3,\,D_0 + D_1,\,D_0 + D_3\}$, $-(K_X + D)$ is ample if and only if $r = 0$.
(3) If $D \in \{D_2,\,D_2 + D_1,\,D_2 + D_3\}$, $-(K_X + D)$ is ample for any $r \in \mathbb{N}$.
(4) If $D \in \{D_0 + D_2,\,D_1 + D_3\}$, $-(K_X + D)$ is not ample for any $r \in \mathbb{N}$.

Proof. As $-K_X = D_0 + D_1 + D_2 + D_3$ and $D_1 \sim_{\text{lin}} D_3$, $D_2 \sim_{\text{lin}} D_0 - r\,D_3$, we get the following linear equivalence of divisors:

$$\begin{aligned}
&-K_X \sim_{\text{lin}} 2D_0 + (2-r)D_3 && -(K_X + D_0 + D_2) \sim_{\text{lin}} 2D_3\\
&-(K_X + D_0) \sim_{\text{lin}} D_0 + (2-r)D_3 && -(K_X + D_0 + D_3) \sim_{\text{lin}} D_0 + (1-r)D_3\\
&-(K_X + D_2) \sim_{\text{lin}} D_0 + 2D_3 && -(K_X + D_2 + D_3) \sim_{\text{lin}} D_0 + D_3\\
&-(K_X + D_3) \sim_{\text{lin}} 2D_0 + (1-r)D_3 && -(K_X + D_1 + D_3) \sim_{\text{lin}} 2D_0 - rD_3
\end{aligned}$$

If $a_1 = a_2 = 0$, the condition (6) becomes $a_0 > 0$ and $a_3 > 0$. This allows us to conclude. ☐

Remark 3.4. Propositions 3.2 and 3.3 give Theorem 1.1.

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