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Faisceaux équivariants stables sur les variétés toriques

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INTRODUCTION

1.1. Français

La notion de *stabilité au sens de la pente*[1] a été introduite par Mumford [30] dans sa construction d'un schéma quasi-projectif décrivant l'espace de modules des fibrés vectoriels d'un rang donné sur une courbe complexe. Cette notion de stabilité a été généralisée en dimension supérieure par Takemoto [36]. On dit qu'un fibré vectoriel, ou plus généralement un faisceau cohérent sans torsion $\mathscr{E}$ sur une variété projective complexe X est *stable* (resp. *semistable*) par rapport à une polarisation L, si pour tout sous-faisceau cohérent propre $\mathscr{F}$ de $\mathscr{E}$ tel que $0 < \mathrm{rg}(\mathscr{F}) < \mathrm{rg}(\mathscr{E})$, on a $\mu_L(\mathscr{F}) < \mu_L(\mathscr{E})$ (resp. $\mu_L(\mathscr{F}) \leq \mu_L(\mathscr{E})$) où la *pente* $\mu_L(\mathscr{E})$ de $\mathscr{E}$ par rapport à L est donnée par la formule

$$\mu_L(\mathscr{E}) = \frac{c_1(\mathscr{E}) \cdot L^{n-1}}{\mathrm{rg}(\mathscr{E})}.$$

D'après Hartshorne [13], les faisceaux réflexifs peuvent être vus comme des fibrés vectoriels légèrement singuliers ; de plus leur étude permet de mieux comprendre les fibrés vectoriels. De ce fait, étudier la stabilité des faisceaux réflexifs présente un grand intérêt. Compte tenue de la difficulté qui existe dans l'étude de la stabilité des faisceaux réflexifs, nous nous intéressons au cas de la catégorie des faisceaux réflexifs équivariants sur les variétés toriques normales en raison de leurs descriptions combinatoires.

On rappelle qu'une variété torique de dimension n est une variété irréductible X contenant un tore $T \simeq (\mathbb{C}^*)^n$ comme ouvert de Zariski dense et telle que l'action de T sur lui-même par multiplication s'étende en une action algébrique sur X (cf. Section 2.1.1). Une variété torique normale X est une variété qui peut être décrite à partir d'un éventail Σ de cônes polyédraux saillants dans $N \otimes_{\mathbb{Z}} \mathbb{R}$ où N est un réseau. On note $\Sigma(1)$ l'ensemble des cônes de dimension 1 de Σ et $u_\rho \in N$ l'élément primitif engendrant $\rho \in \Sigma(1)$. Enfin, un faisceau $\mathscr{E}$ sur une variété torique normale est T-équivariant (ou équivariant) s'il possède un isomorphisme $\Phi : \theta^*\mathscr{E} \longrightarrow \mathrm{pr}_2^*\mathscr{E}$ qui satisfait une relation cocyclique (2.14) où $\theta : T \times X \longrightarrow X$ est l'action de T sur X et $\mathrm{pr}_2 : T \times X \longrightarrow X$ la projection sur le second facteur.

Klyachko [23] a donné une description complète des fibrés vectoriels équivariants sur les variétés toriques en termes de familles de filtrations d'espaces vectoriels. Cette description a été étendue au cas des faisceaux réflexifs équivariants par Perling [33]. Le fait de supposer le faisceau équivariant apporte de nombreuses simplifications dans l'étude de sa stabilité. En effet, si $\mathscr{E}$ est un faisceau réflexif équivariant sur une variété torique projective normale X, d'après [25, Proposition 4.13], il est suffisant d'étudier la stabilité en ne considérant que des sous-faisceaux

[1] La stabilité au sens de la pente est aussi appelée stabilité au sens de Mumford–Takemoto

réflexifs équivariants saturés. (On dit qu'un sous-faisceau cohérent $\mathscr{F}$ de $\mathscr{E}$ est *saturé* dans $\mathscr{E}$ si le faisceau quotient $\mathscr{E}/\mathscr{F}$ est sans-torsion.) De plus, pour toute polarisation L de X, l'ensemble

$$\{\mu_L(\mathscr{F}) : \mathscr{F} \text{ un sous-faisceau réflexif équivariant et saturé de } \mathscr{E}\} \tag{1.1}$$

est fini.

En utilisant le fait que le fibré tangent d'une variété torique normale est équivariant, Hering-Nill-Süss [14] et Dasgupta-Dey-Khan [4] ont étudié la stabilité du fibré tangent sur les variétés toriques projectives lisses de rang de Picard un et deux. En s'inspirant de la philosophie de Iitaka, nous avons étendu les résultats de [4] et [14] aux cas des paires logarithmiques équivariantes. D'après la Proposition 3.1.3, le faisceau tangent logarithmique $\mathscr{T}_X(-\log D)$ est un sous-faisceau équivariant du faisceau tangent $\mathscr{T}_X$ si et seulement si D est un diviseur de X invariant par l'action du tore et réduit. Donc,

$$D = \sum_{\rho\in\Delta} D_\rho$$

où $\Delta \subseteq \Sigma(1)$ et D_ρ le diviseur premier et invariant de X déterminé par $\rho \in \Sigma(1)$.

Théorème 1.1.1 (Theorem 3.1.5). *Soit $\Delta \subseteq \Sigma(1)$ et $D = \sum_{\rho\in\Delta} D_\rho$ un diviseur réduit de X. La famille de filtrations $\left(E, \{E^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\right)$ du faisceau tangent logarithmique $\mathscr{T}_X(-\log D)$ est donnée par :*

$$E^\rho(j) = \begin{cases} 0 & \text{si } j \le -1 \\ N\otimes_{\mathbb{Z}} \mathbb{C} & \text{si } j \ge 0 \end{cases} \qquad \text{si } \rho \in \Delta$$

et

$$E^\rho(j) = \begin{cases} 0 & \text{si } j \le -2 \\ \mathrm{vect}(u_\rho) & \text{si } j = -1 \\ N\otimes_{\mathbb{Z}} \mathbb{C} & \text{si } j \ge 0 \end{cases} \qquad \text{si } \rho \notin \Delta\,.$$

Si $\Delta = \Sigma(1)$, le faisceau $\mathscr{T}_X(-\log D)$ est isomorphe au faisceau trivial de rang $\dim(X)$ et si $\Delta = \varnothing$, $\mathscr{T}_X(-\log D)$ est le faisceau tangent. En utilisant le Théorème 1.1.1 et le fait que $|\Sigma(1)| = \dim(X) + \mathrm{rg}(\mathrm{Cl}(X))$, on montre que :

Proposition 1.1.2. *Si $\mathrm{rg}(\mathrm{Cl}(X)) + 1 \le |\Delta| \le |\Sigma(1)| - 1$, alors pour toute polarisation L, le faisceau tangent logarithmique $\mathscr{T}_X(-\log D)$ est instable par rapport à L.*

De ce fait, il suffit d'étudier la stabilité du faisceau tangent logarithmique $\mathscr{T}_X(-\log D)$ lorsque $|\Delta| \le \mathrm{rg}(\mathrm{Cl}(X))$. Dans ce mémoire, nous étudions le cas où X est une variété projective torique lisse telle que $\mathrm{rg}\,\mathrm{Cl}(X) \in \{1, 2\}$. L'espace projective complexe $\mathbb{P}^n$ est la seule variété torique lisse de rang de Picard un.

Proposition 1.1.3. *Soit D une section hyperplane de $\mathbb{P}^n$ invariante par l'action du tore. Alors, le faisceau tangent logarithmique $\mathscr{T}_{\mathbb{P}^n}(-\log D)$ est polystable par rapport à $\mathscr{O}_{\mathbb{P}^n}(1)$.*

D'après [22, Theorem 1], toute variété torique lisse de rang de Picard deux est de la forme $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^r \mathscr{O}_{\mathbb{P}^s}(a_i))$ avec $r, s \in \mathbb{N}^*$ et $a_1, \ldots, a_r \in \mathbb{N}$ tels que $a_1 \le \ldots \le a_r$. Notons $\pi : X \longrightarrow \mathbb{P}^s$ l'application projection et $\mathscr{V}$ le fibré vectoriel associé au faisceau localement libre

$$\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(-a_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^s}(-a_r).$$

Les diviseurs irréductibles et invariants de X sont donnés par

$$\begin{cases} D_{w_j} = \pi^{-1}(\{(z_0 : \ldots : z_s) \in \mathbb{P}^s : z_j = 0\}) & \text{pour } 0 \le j \le s \\ D_{v_i} = \{s_i = 0\} & \text{pour } 0 \le i \le r \end{cases}$$

où les $\{s_i = 0\}$ sont des sections hyperplanes relatives associées aux sous-fibrés en droites de $\mathscr{V}^\vee$. Si $a_1 = \ldots = a_r = 0$, on montre que :

Theorem 1.1.4 (Theorem 3.3.1). *Soit $i \in \{0, \dots, r\}$ et $j \in \{0, \dots, s\}$. Alors :*

1. *$\mathscr{T}_X(-\log D_{v_i})$ est polystable par rapport à $\pi^*\mathscr{O}_{\mathbb{P}^s}(\alpha) \otimes \mathscr{O}_X(\beta)$ si et seulement si $\frac{\alpha}{\beta} = \frac{s+1}{r}$;*
2. *$\mathscr{T}_X(-\log D_{w_j})$ est polystable par rapport à $\pi^*\mathscr{O}_{\mathbb{P}^s}(\alpha) \otimes \mathscr{O}_X(\beta)$ si et seulement si $\frac{\alpha}{\beta} = \frac{s}{r+1}$;*
3. *$\mathscr{T}_X(-\log(D_{v_i} + D_{w_j}))$ est polystable par rapport à $\pi^*\mathscr{O}_{\mathbb{P}^s}(\alpha) \otimes \mathscr{O}_X(\beta)$ si et seulement si $\frac{\alpha}{\beta} = \frac{s}{r}$.*

Lorsque $a_r \geq 1$, on a obtenu la classification suivante des paires (X, D) telles que le faisceau $\mathscr{T}_X(-\log D)$ est (semi)stable. Plus précisément, on donne les valeurs de ν pour lesquelles $\mathscr{E} = \mathscr{T}_X(-\log D)$ est (semi)stable par rapport à $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ dans les tableaux 3.2, 3.3, 3.4 et leurs références.

Theorem 1.1.5 (Theorem 3.3.5). *Soit $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(a_1) \oplus \dots \oplus \mathscr{O}_{\mathbb{P}^s}(a_r))$ avec $(a_1, \dots, a_r) \neq (0, \dots, 0)$ et D un diviseur réduit et invariant de X. Alors :*

1. *Il existe une polarisation L telle que $\mathscr{T}_X(-\log D)$ est stable par rapport à L si et seulement si :*
   i. *$(a_1, \dots, a_r) = (0, \dots, 0, 1)$ et $D = D_{v_r}$, ou*
   ii. *$a_1 = \dots = a_r$ vérifie $(r-1)a_r < (s+1)$ et $D = D_{v_0}$.*
2. *Il existe une polarisation L telle que $\mathscr{T}_X(-\log D)$ est polystable par rapport à L si et seulement si :*
   i. *$a_1 = \dots = a_r$ vérifie $(r-1)a_r < s$ et*
   $$D \in \{D_{v_0} + D_{w_j} : 0 \leq j \leq s\} \cup \{D_{v_0} + D_{v_i} : 1 \leq i \leq r\},$$
   ii. *ou $1 \leq a_1 < a_2 = \dots = a_r$ et $D = D_{v_0} + D_{v_1}$ avec $\ell(s) > 0$ où l'application $\ell : \mathbb{N}^* \longrightarrow \mathbb{R}$ est définie par*
   $$\ell(p) = \sum_{j=0}^{p-1} \binom{j+r-2}{j} \left(1 - \frac{a_r(r-2)}{j+1}\right) \left(\frac{a_r}{a_1}\right)^j - a_1.$$
3. *Dans les autres cas, le faisceau $\mathscr{T}_X(-\log D)$ est instable par rapport à toute polarisation.*

Dans l'article [27], Maeda a classifié les surfaces log del Pezzo et les variétés log-Fano de dimension 3. Dans la Proposition 3.4.4, nous donnons une preuve combinatoire de ce résultat pour les surfaces toriques. En utilisant la Proposition 3.4.4 et le Théorème 1.2.5, nous obtenons :

Proposition 1.1.6. *Soit $r \in \mathbb{N}$ et $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(r))$ munie de la projection $\pi : X \longrightarrow \mathbb{P}^1$. Soit D', D_0, D_2 des diviseurs invariants tels que :*

- *D' est une fibre de π ;*
- *D_0 et D_2 sont des sections vérifiant $D_0 \cdot D_0 = r$ et $D_2 \cdot D_2 = -r$.*

Alors,

1. *si $r = 0$ et $D \in \{D', D_0, D_2, D_0 + D', D_2 + D'\}$, $\mathscr{T}_X(-\log D)$ est polystable par rapport à $-(K_X + D)$;*
2. *si $r = 1$, $\mathscr{T}_X(-\log D_0)$ est stable par rapport à $-(K_X + D_0)$.*
3. *si $r \geq 1$ et $D \in \{D_2, D_2 + D'\}$, $\mathscr{T}_X(-\log D)$ est instable par rapport à $-(K_X + D)$.*

À partir de faisceaux stables, une question naturelle est de savoir comment se comporte la stabilité à travers certaines opérations comme les pullbacks. Dans le cas des immersions, le théorème de Mehta et Ramanathan nous dit :

Théorème 1.1.7 ([28, Theorem 4.3]). *Soit X une variété projective lisse de dimension n, H une polarisation et $V_{(k)} = D_1 \cap \dots \cap D_j$ une sous-variété obtenue comme intersection complète générique d'éléments $D_i \in |kH|$ pour k suffisamment grand. Si $\mathscr{E}$ est un faisceau cohérent sans torsion semistable (resp. stable) par rapport à H, alors il existe $k_0 \in \mathbb{N}^*$, tel que pour tout $k \geq k_0$, la restriction $\mathscr{E}_{|V_{(k)}}$ est semistable (resp. stable) par rapport à $H_{|V_{(k)}}$.*

Dans ce mémoire, nous étudions le problème des pullbacks de faisceaux réflexifs semistables à travers des fibrations dans le cas équivariant de la géométrie torique. Un application propre $\pi : X' \longrightarrow X$ est une *fibration* si $\pi_*\mathscr{O}_{X'} = \mathscr{O}_X$. Soit $\pi : X' \longrightarrow X$ une fibration entre deux variétés toriques projectives $\mathbb{Q}$-factorielles et $\mathscr{E}$ un faisceau réflexif équivariant sur X. On note $\mathscr{E}' = (\pi^*\mathscr{E})^{\vee\vee}$ l'enveloppe réflexive de $\pi^*\mathscr{E}$ et pour $\mathscr{F} \subsetneq \mathscr{E}$ un sous-faisceau, on note $(\pi^*\mathscr{F})^{\mathrm{sat}}$ la saturation de $(\pi^*\mathscr{F})^{\vee\vee}$ dans $\mathscr{E}'$, c'est le noyau de la surjection

$$\mathscr{E}' \longrightarrow \mathscr{E}'/(\pi^*\mathscr{F})^{\vee\vee} \longrightarrow (\mathscr{E}'/(\pi^*\mathscr{F})^{\vee\vee})/\operatorname{Tor}(\mathscr{E}'/(\pi^*\mathscr{F})^{\vee\vee})$$

(cf. [16, Definition 1.1.5]). Soit L un diviseur ample sur X et L' un diviseur π-ample. Pour $\varepsilon \in \mathbb{Q}_{>0}$ suffisamment petit, $L_\varepsilon = \pi^*L + \varepsilon L'$ définit un $\mathbb{Q}$-diviseur ample sur X'. En suivant la terminologie utilisée en géométrie différentielle, le $\mathbb{Q}$-diviseur ample L_ε est appelé *polarisation adiabatique*. La pente de $\mathscr{E}'$ (resp. $(\pi^*\mathscr{F})^{\mathrm{sat}}$) par rapport à L_ε est un polynôme en ε tel que le coefficient du plus petit exposant du développement en ε est donné par $\mu_L(\mathscr{E})$ (resp. $\mu_L(\mathscr{F})$). À l'aide de (1.1) il est facile de montrer : si $\mathscr{E}$ est stable (resp. instable) par rapport à L, alors il existe $\varepsilon_0 \in \mathbb{Q}_{>0}$ tel que pour tout $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, $\mathscr{E}'$ est stable (resp. instable) par rapport à L_ε.

Le résultat principal sur les pullbacks de faisceaux le long des fibrations que nous montrons concerne le cas strictement semistable. Pour deux faisceaux cohérents $\mathscr{F}_1$ et $\mathscr{F}_2$ sur X' on écrit $\mu_0(\mathscr{F}_1) < \mu_0(\mathscr{F}_2)$ (resp. $\mu_0(\mathscr{F}_1) \leq \mu_0(\mathscr{F}_2)$ ou bien $\mu_0(\mathscr{F}_1) = \mu_0(\mathscr{F}_2)$) si le coefficient du plus petit exposant du développement en ε de $\mu_{L_\varepsilon}(\mathscr{F}_2) - \mu_{L_\varepsilon}(\mathscr{F}_1)$ est strictement positif (resp. positif ou nul ou bien nul). Si $\mathscr{E}$ est un faisceau cohérent sans torsion strictement semistable par rapport à L sur X, il existe une filtration de Jordan-Hölder

$$0 = \mathscr{E}_1 \subseteq \mathscr{E}_2 \subseteq \ldots \subseteq \mathscr{E}_\ell = \mathscr{E}$$

par des sous-faisceaux cohérents semistables avec des quotients stables et de même pente que $\mathscr{E}$. On note $\operatorname{Gr}_L(\mathscr{E}) := \bigoplus_{i=1}^{\ell-1} \mathscr{E}_{i+1}/\mathscr{E}_i$ le gradué de $\mathscr{E}$ et $\mathfrak{E}$ l'ensemble des sous-faisceaux saturés $\mathscr{F} \subseteq \mathscr{E}$ provenant de la filtration de Jordan-Hölder de $\mathscr{E}$.

Théorème 1.1.8 (Theorem 4.1.9)**.** *Soit $\mathscr{E}$ un faisceau équivariant localement libre et strictement semistable sur la variété polarisée (X, L) tel que son objet gradué soit localement libre. Alors il existe $\varepsilon_0 \in \mathbb{Q}_{>0}$ tel que pour tout $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, le faisceau $\mathscr{E}'$ sur X' est*

1. *stable par rapport à L_ε si et seulement si pour tout $\mathscr{F} \in \mathfrak{E}$, $\mu_0((\pi^*\mathscr{F})^{\vee\vee}) < \mu_0(\mathscr{E}')$,*
2. *strictement semistable par rapport à L_ε si et seulement si pour tout $\mathscr{F} \in \mathfrak{E}$, $\mu_0((\pi^*\mathscr{F})^{\vee\vee}) \leq \mu_0(\mathscr{E}')$,*
3. *instable par rapport à L_ε si et seulement s'il existe $\mathscr{F} \in \mathfrak{E}$ avec $\mu_0((\pi^*\mathscr{F})^{\vee\vee}) > \mu_0(\mathscr{E}')$.*

Remarque 1.1.9. Dans le cas où $\pi : X' \longrightarrow X$ est une fibration torique localement triviale, on montre que les hypothèses sur $\mathscr{E}$ et $\operatorname{Gr}_L(\mathscr{E})$ d'être localement libres ne sont pas nécessaires.

Un autre exemple de fibration qui est étudié est celui des éclatements. Soit X une variété projective lisse et Z une sous-variété irréductible lisse de dimension ℓ telle que $\ell \leq \dim(X) - 2$. On note $\pi : X' \longrightarrow X$ l'éclatement de X le long de Z et D_0 son diviseur exceptionnel. Pour L une polarisation sur X et $\varepsilon \in \mathbb{Q}_{>0}$ petit, on considère la polarisation L_ε sur X' définie par $L_\varepsilon = \pi^*L - \varepsilon D_0$.

Proposition 1.1.10. *Soit Z un ensemble de points invariants d'une variété torique X et $\pi : X' \longrightarrow X$ l'éclatement de X le long de Z. Soit $\mathscr{E}$ un faisceau réflexif équivariant et strictement semistable sur la variété polarisée (X, L). Alors, il existe $\varepsilon_0 \in \mathbb{Q}_{>0}$ tel que pour tout $\varepsilon \in]0, \varepsilon_0[$, le faisceau $\mathscr{E}' = (\pi^*\mathscr{E})^{\vee\vee}$ est*

1. *semistable par rapport à L_ε si et seulement si pour tout $\mathscr{F} \in \mathfrak{E}$, $(\pi^*\mathscr{F})^{\vee\vee}$ est saturé dans $\mathscr{E}'$;*
2. *instable dans l'autre cas.*

Le résultat [9, Proposition 5.1] est plus général que la proposition ci-dessus puisqu'il considère le cas des faisceaux cohérents sans-torsion sur une variété projective normale (non nécessairement torique). En revanche, la Proposition 1.1.10 donne plus d'informations lorsque $\mathrm{Gr}_L(\mathscr{E})$ n'est pas localement libre. Si l'on suppose $\ell \geq 1$, pour tout faisceau réflexif $\mathscr{E}$ sur une variété projective lisse X, on montre que

$$\mu_{L_\varepsilon}((\pi^*\mathscr{E})^{\vee\vee}) = \mu_L(\mathscr{E}) - \binom{n-1}{\ell-1}\mu_{L_{|Z}}(\mathscr{E}_{|Z})\varepsilon^{n-\ell} + O(\varepsilon^{n-\ell+1}). \tag{1.2}$$

En particulier, si X est une variété torique, Z une sous variété invariante de X et $\mathscr{E}$ un faisceau équivariant strictement semistable par rapport à L tel que pour tout $\mathscr{F} \in \mathfrak{E}$, $(\pi^*\mathscr{F})^{\vee\vee}$ est saturé dans $\mathscr{E}' = (\pi^*\mathscr{E})^{\vee\vee}$ et

$$\mu_{L_{|Z}}(\mathscr{E}_{|Z}) < \mu_{L_{|Z}}(\mathscr{F}_{|Z}) ,$$

alors il existe $\varepsilon_0 \in \mathbb{Q}_{>0}$ tel que pour tout $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, $\mathscr{E}'$ est stable par rapport à L_ε. Dans le cas où $\ell = 1$, nous avons :

Théorème 1.1.11 (Theorem 4.2.10)**.** *Soit X une variété torique projective lisse et $\pi : X' \longrightarrow X$ l'éclatement de X le long d'une courbe invariante Z. Si $\mathscr{E}$ est un faisceau réflexif équivariant sur X et strictement semistable par rapport à L, alors il existe $\varepsilon_0 \in \mathbb{Q}_{>0}$, tel que pour tout $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, le faisceau $\mathscr{E}' = (\pi^*\mathscr{E})^{\vee\vee}$ est :*

1. *stable par rapport à L_ε si et seulement si pour tout $\mathscr{F} \in \mathfrak{E}$, $(\pi^*\mathscr{F})^{\vee\vee}$ est saturé dans $\mathscr{E}'$ et $\dfrac{c_1(\mathscr{E}) \cdot Z}{\operatorname{rk} \mathscr{E}} < \dfrac{c_1(\mathscr{F}) \cdot Z}{\operatorname{rk} \mathscr{F}}$;*
2. *semistable par rapport à L_ε si et seulement si pour tout $\mathscr{F} \in \mathfrak{E}$, $(\pi^*\mathscr{F})^{\vee\vee}$ est saturé dans $\mathscr{E}'$ et $\dfrac{c_1(\mathscr{E}) \cdot Z}{\operatorname{rk} \mathscr{E}} \leq \dfrac{c_1(\mathscr{F}) \cdot Z}{\operatorname{rk} \mathscr{F}}$;*
3. *instable dans les autres cas.*

Remark 1.1.12. Grâce à ce théorème, nous donnons dans la Section 4.2.5 un exemple explicite de faisceau strictement semistable, à savoir le faisceau tangent de $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1}^{\oplus r} \oplus \mathscr{O}_{\mathbb{P}^1}(1))$, qui devient stable ou instable lorsqu'on considère son pullback le long de l'éclatement d'une courbe.

Ces résultats sur la stabilité par passage aux éclatements ont une application à la résolution de singularités des faisceaux. Le théorème d'Hironaka [15, Main Theorem II] nous dit : Pour un faisceau réflexif $\mathscr{E}_0 := \mathscr{E}$ sur une variété projective lisse $X_0 := X$, il existe une suite finie d'éclatements le long de centres invariants $\pi_i : X_i \longrightarrow X_{i-1}$ pour $1 \leq i \leq p$ telle que : si l'on pose $\mathscr{E}_i = (\pi_i^*\mathscr{E}_{i-1})^{\vee\vee}$, alors $\mathscr{E}' := \mathscr{E}_p$ est localement libre sur $X' := X_p$. On note $\pi : X_p \longrightarrow X$ l'application entre X_p et X et $S := X(\mathscr{E})_{\mathrm{sing}}$ le lieu singulier de $\mathscr{E}$ sur X.

Corollaire 1.1.13. *Soit $\mathscr{E}$ un faisceau équivariant stable sur la variété torique polarisée (X, L), alors il existe $\varepsilon_0 \in \mathbb{Q}_{>0}$ tel que pour tout $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, $\mathscr{E}'$ est stable par rapport à $\pi^*L - \varepsilon E$ où $E = \pi^{-1}(S)$.*

Le nombre p donné ci-dessus n'est pas explicite. Une question naturelle serait donc de trouver une borne explicite sur p. Dans ce mémoire, nous décrivons le lieu singulier des faisceaux réflexifs équivariants. Généralement, si $\mathscr{E}$ est un faisceau réflexif sur une variété projective complexe X, son lieu singulier $X(\mathscr{E})_{\mathrm{sing}}$ est un sous-ensemble fermé dans X pour la topologie de Zariski et de codimension au moins 3. Dans le cas où X est torique et $\mathscr{E}$ équivariant, il est facile de montrer que $X(\mathscr{E})_{\mathrm{sing}}$ est une union fini de fermeture d'orbites de X. Plus précisement, si Σ est l'éventail de X, il existe $\tau_1, \ldots, \tau_r \in \Sigma$ tels que $\dim(\tau_i) \geq 3$ et

$$X(\mathscr{E})_{\mathrm{sing}} = \bigcup_{i=1}^{r} V(\tau_i)$$

où $V(\tau_i)$ est la fermeture de l'orbite associée au cône τ_i. On montre ensuite qu'il est possible de prescrire des singularités sur un faisceau.

Théorème 1.1.14 (Theorem 5.1.6)**.** *Soit X une variété torique lisse d'éventail Σ. Soit $\tau_1, \dots, \tau_m \in \Sigma$ vérifiant $\dim(\tau_i) \geq 3$ et tels que pour tout $i \neq j$, τ_i n'est pas une face propre de τ_j. Alors, il existe un faisceau réflexif équivariant $\mathscr{E}$ de rang $\sum_{i=1}^{r} \dim(\tau_i) - m$ sur X tel que*

$$X(\mathscr{E})_{\text{sing}} = \bigcup_{i=1}^{m} V(\tau_i).$$

La construction du faisceau $\mathscr{E}$ est explicite et s'obtient comme généralisation de l'exemple d'Hartshorne [13, Example 1.9.1].

Plan du document. Le manuscrit est construit de la façon suivante :

- Dans le Chapitre 2, on rappelle les notions de base sur les variétés toriques, les faisceaux équivariants et aussi les notions sur la stabilité. Ce chapitre s'appuie sur [2], [33] et [36].
- Dans le Chapitre 3 nous étudions la stabilité des faisceaux tangents logarithmiques équivariants. Nous montrons les Théorèmes 1.1.1, 1.1.4, 1.1.5 et les Propositions 1.1.2, 1.1.3. Les résultat de ce chapitre ont donné lieu à l'article [Nap21].
- Le Chapitre 4 présente les notions de stabilité des pullbacks de faisceaux le long des fibrations. Dans ce chapitre, nous montrons les Théorèmes 1.1.8 et 1.1.11. Nous démontrons également la formule (1.2). Ces résultats sont basés sur l'article [NT22].
- Enfin dans le Chapitre 5, on étudie le lieu singulier des faisceaux réflexifs équivariants sur les variétés toriques. Nous montrons le Théorème 1.1.14.

1.2. English

The notion of *slope stability*[2] was first introduced by Mumford [30] in his construction of moduli spaces of vector bundles over a curve. This notion was generalized in higher dimension by Takemoto [36]. A vector bundle, or more generally a torsion-free sheaf $\mathscr{E}$ on a complex projective variety X is said to be slope stable (resp. semistable) with respect to a polarization L, if for any proper coherent subsheaf $\mathscr{F}$ of $\mathscr{E}$ with $0 < \operatorname{rk}(\mathscr{F}) < \operatorname{rk}(\mathscr{E})$, one has $\mu_L(\mathscr{F}) < \mu_L(\mathscr{E})$ (resp. $\mu_L(\mathscr{F}) \leq \mu_L(\mathscr{E})$) where the *slope* $\mu_L(\mathscr{E})$ of $\mathscr{E}$ with respect to L is given by

$$\mu_L(\mathscr{E}) = \frac{c_1(\mathscr{E}) \cdot L^{n-1}}{\operatorname{rk}(\mathscr{E})}.$$

According to Hartshorne [13], reflexive sheaves can be seen as vector bundles with singularities and their study gives a better description of vector bundles. This gives us a reason to study reflexive sheaves. As the study of stability is a difficult problem, we are interested in the category of torus equivariant reflexive sheaves over normal toric varieties according to their combinatorial data.

We recall that an n-dimensional toric variety is an irreducible variety X containing a torus $T \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T on itself by multiplication extends to an algebraic action of T on X (cf. Section 2.1.1). According to [2, Corollary 3.1.8], a normal toric variety X comes from a fan Σ of strongly convex polyhedral cones in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ where N is a lattice. We denote by $\Sigma(1)$ the set of one dimensional cones of Σ and by $u_\rho \in N$ the

[2] Slope stability is also called Mumford–Takemoto–stability

minimal generator of $\rho \in \Sigma(1)$. We say that a reflexive sheaf $\mathscr{E}$ on a normal toric variety X is T-equivariant (or equivariant for short) if it is equipped with an isomorphism $\Phi : \theta^*\mathscr{E} \longrightarrow \mathrm{pr}_2^*\mathscr{E}$ which satisfies some cocyle condition (2.14) where $\theta : T \times X \longrightarrow X$ is the action of T on X and $\mathrm{pr}_2 : T \times X \longrightarrow X$ the projection onto the second factor.

Klyachko [23] gave a description of torus equivariant vector bundles over toric varieties in terms of families of filtrations of vector spaces. This classification was extended to the case of torsion-free equivariant coherent sheaves by Perling [33]. The equivariant structure on a sheaf gives a lot of simplifications in the study of its stability. If $\mathscr{E}$ is an equivariant reflexive sheaf on a normal projective toric variety X, according to [25, Proposition 4.13], it is enough to test slope inequalities for equivariant and reflexive saturated subsheaves. (A coherent subsheaf $\mathscr{F}$ of $\mathscr{E}$ is *saturated* if the quotient sheaf $\mathscr{E}/\mathscr{F}$ is torsion-free.) Moreover, for any polarization L on X, the set

$$\{\mu_L(\mathscr{F}) : \mathscr{F} \text{ is an equivariant reflexive and saturated subsheaf of } \mathscr{E}\} \tag{1.3}$$

is finite.

By using the equivariant structure of the tangent bundle, Dasgupta-Dey-Khan in [4] and Hering-Nill-Süss in [14] studied slope-stability of the tangent bundle of smooth projective toric varieties of Picard rank one or two. Inspired by Iitaka's philosophy, we extend the results of [4] and [14] to the case of equivariant log pairs (X, D). By Proposition 3.1.3, the logarithmic tangent sheaf $\mathscr{T}_X(-\log D)$ is an equivariant subsheaf of the tangent sheaf if and only if the reduced divisor D is torus invariant. Hence,

$$D = \sum_{\rho \in \Delta} D_\rho$$

where $\Delta \subseteq \Sigma(1)$ and D_ρ is a prime and torus invariant divisor of X corresponding to the ray $\rho \in \Sigma(1)$.

Theorem 1.2.1 (Theorem 3.1.5). *Let $\Delta \subseteq \Sigma(1)$ and $D = \sum_{\rho\in\Delta} D_\rho$ be a reduced divisor of X. The family of filtrations $\left(E, \{E^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\right)$ of the logarithmic tangent sheaf $\mathscr{T}_X(-\log D)$ is given by*

$$E^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } j \geq 0 \end{cases} \qquad \text{if } \rho \in \Delta$$

and by

$$E^\rho(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ \mathrm{Span}(u_\rho) & \text{if } j = -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } j \geq 0 \end{cases} \qquad \text{if } \rho \notin \Delta\,.$$

If $\Delta = \Sigma(1)$, then $\mathscr{T}_X(-\log D)$ is isomorphic to the trivial sheaf of rank n and if $\Delta = \varnothing$, then $\mathscr{T}_X(-\log D)$ is the tangent sheaf $\mathscr{T}_X$. By Theorem 1.2.1 and the fact that $|\Sigma(1)| = \dim(X) + \mathrm{rk}(\mathrm{Cl}(X))$, we show that:

Proposition 1.2.2. *If $1 + \mathrm{rk}(\mathrm{Cl}(X)) \leq |\Delta| \leq |\Sigma(1)| - 1$, then for any polarization L, the logarithmic tangent sheaf $\mathscr{T}_X(-\log D)$ is unstable with respect to L.*

According to this proposition, it is therefore sufficient to study the stability of $\mathscr{T}_X(-\log D)$ when $|\Delta| \leq \mathrm{rk}(\mathrm{Cl}(X))$. Thus, in this thesis we study the case where X is smooth, $\mathrm{rk}\,\mathrm{Cl}(X) \in \{1, 2\}$ and $1 \leq |\Delta| \leq \mathrm{rk}\,\mathrm{Cl}(X)$. Note that the only smooth projective toric variety with Picard number one is the projective space $\mathbb{P}^n$.

Proposition 1.2.3. *Let D be an invariant hyperplane section of $\mathbb{P}^n$. Then, the logarithmic tangent sheaf $\mathscr{T}_{\mathbb{P}^n}(-\log D)$ is polystable with respect to $\mathscr{O}_{\mathbb{P}^n}(1)$.*

By [22, Theorem 1], every smooth toric variety of Picard rank two is of the form $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^r \mathscr{O}_{\mathbb{P}^s}(a_i))$ with $r, s \in \mathbb{N}^*$ and $a_1, \ldots, a_r \in \mathbb{N}$ such that $a_1 \leq \ldots \leq a_r$. We denote by $\pi : X \longrightarrow \mathbb{P}^s$ the projection map. Let $\mathscr{V}$ be a vector bundle associated to the locally free sheaf

$$\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(-a_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^s}(-a_r).$$

Then the irreducible invariant divisors of X are given by

$$\begin{cases} D_{w_j} = \pi^{-1}(\{(z_0 : \ldots : z_s) \in \mathbb{P}^s : z_j = 0\}) & \text{for } 0 \leq j \leq s \\ D_{v_i} = \{s_i = 0\} & \text{for } 0 \leq i \leq r \end{cases}$$

where the $\{s_i = 0\}$ are the relative hyperplane sections associated to the line subbundles of $\mathscr{V}^\vee$. If $a_1 = \ldots = a_r = 0$, we show that:

Theorem 1.2.4 (Theorem 3.3.1). *Let $i \in \{0, \ldots, r\}$ and $j \in \{0, \ldots, s\}$. Then:*

1. *$\mathscr{T}_X(-\log D_{v_i})$ is polystable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\alpha) \otimes \mathscr{O}_X(\beta)$ if and only if $\frac{\alpha}{\beta} = \frac{s+1}{r}$;*
2. *$\mathscr{T}_X(-\log D_{w_j})$ is polystable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\alpha) \otimes \mathscr{O}_X(\beta)$ if and only if $\frac{\alpha}{\beta} = \frac{s}{r+1}$;*
3. *$\mathscr{T}_X(-\log(D_{v_i} + D_{w_j}))$ is polystable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\alpha) \otimes \mathscr{O}_X(\beta)$ if and only if $\frac{\alpha}{\beta} = \frac{s}{r}$.*

When $a_r \geq 1$, we get the following classification on pairs (X, D) such that $\mathscr{T}_X(-\log D)$ is (semi)stable. More precisely, we give the values of ν for which $\mathscr{E} = \mathscr{T}_X(-\log D)$ is (semi)stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ in the Tables 3.2, 3.3, 3.4 and the references therein.

Theorem 1.2.5 (Theorem 3.3.5). *Let $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(a_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^s}(a_r))$ with $(a_1, \ldots, a_r) \neq (0, \ldots, 0)$ and D a reduced invariant divisor of X. Then:*

1. *There is a polarization L such that $\mathscr{T}_X(-\log D)$ is stable with respect to L if and only if:*
   i. *$(a_1, \ldots, a_r) = (0, \ldots, 0, 1)$ and $D = D_{v_r}$, or*
   ii. *$a_1 = \ldots = a_r$ with $(r-1)a_r < (s+1)$ and $D = D_{v_0}$.*
2. *There is a polarization L such that $\mathscr{T}_X(-\log D)$ is polystable with respect to L if and only if:*
   i. *$a_1 = \ldots = a_r$ with $(r-1)a_r < s$ and*
   $$D \in \{D_{v_0} + D_{w_j} : 0 \leq j \leq s\} \cup \{D_{v_0} + D_{v_i} : 1 \leq i \leq r\},$$
   ii. *or $1 \leq a_1 < a_2 = \ldots = a_r$ and $D = D_{v_0} + D_{v_1}$ with $\ell(s) > 0$ where $\ell : \mathbb{N}^* \longrightarrow \mathbb{R}$ is the map given by*
   $$\ell(p) = \sum_{j=0}^{p-1} \binom{j+r-2}{j} \left(1 - \frac{a_r(r-2)}{j+1}\right) \left(\frac{a_r}{a_1}\right)^j - a_1.$$
3. *Otherwise, the sheaf $\mathscr{T}_X(-\log D)$ is unstable with respect to any polarization.*

Maeda classified log del Pezzo surfaces and log-Fano threefolds in [27]. We give a combinatorial proof of this result for toric surfaces in Proposition 3.4.4. By using Proposition 3.4.4 and Theorem 1.2.5, we get:

Proposition 1.2.6. *Let $r \in \mathbb{N}$ and $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(r))$ with the projection map $\pi : X \longrightarrow \mathbb{P}^1$. Let D', D_0, D_2 be invariant divisors such that:*

- *D' is a fiber of π;*
- *D_0 and D_2 are sections such that $D_0 \cdot D_0 = r$ and $D_2 \cdot D_2 = -r$.*

Then,

1. *if $r = 0$ and $D \in \{D', D_0, D_2, D_0 + D', D_2 + D'\}$, $\mathscr{T}_X(-\log D)$ is polystable with respect to $-(K_X + D)$;*

2. *if $r = 1$, $\mathcal{T}_X(-\log D_0)$ is stable with respect to $-(K_X + D_0)$.*
3. *if $r \geq 1$ and $D \in \{D_2, D_2 + D'\}$, $\mathcal{T}_X(-\log D)$ is unstable with respect $-(K_X + D)$.*

Given stable sheaves, a natural question is to understand how they behave with respect to natural maps such as pullbacks. In the case of immersions, the Metha-Ramanathan theorem says:

Theorem 1.2.7 ([28, Theorem 4.3]). *Let X be an n-dimensional smooth projective variety, H a polarization and $V_{(k)} = D_1 \cap \ldots \cap D_j$ a subvariety obtained as a generic complete intersection of elements $D_i \in |kH|$. If $\mathcal{E}$ is a semistable (resp. stable) torsion-free sheaf with respect to H, then there is $k_0 \in \mathbb{N}^*$ such that: for any $k \geq k_0$, the restriction of $\mathcal{E}_{|V_{(k)}}$ is semistable (resp. stable) with respect to $H_{|V_{(k)}}$.*

In this thesis, we address the problem of pulling-back (semi)stable reflexive sheaves along fibrations, in the equivariant context of toric geometry. A proper morphism $\pi : X' \longrightarrow X$ is a *fibration* if $\pi_* \mathcal{O}_{X'} = \mathcal{O}_X$. Let $\pi : X' \longrightarrow X$ be a fibration between $\mathbb{Q}$-factorial projective toric varieties and $\mathcal{E}$ an equivariant reflexive sheaf on X. We denote by $\mathcal{E}'$ its reflexive pullback $(\pi^*\mathcal{E})^{\vee\vee}$ and for a subsheaf $\mathcal{F} \subsetneq \mathcal{E}$, we denote by $(\pi^*\mathcal{F})^{\mathrm{sat}}$ the saturation of $(\pi^*\mathcal{F})^{\vee\vee}$ in $\mathcal{E}'$, it is the kernel of the surjection

$$\mathcal{E}' \longrightarrow \mathcal{E}'/(\pi^*\mathcal{F})^{\vee\vee} \longrightarrow (\mathcal{E}'/(\pi^*\mathcal{F})^{\vee\vee})/\operatorname{Tor}(\mathcal{E}'/(\pi^*\mathcal{F})^{\vee\vee})$$

(cf. [16, Definition 1.1.5]). Let L be an ample divisor on X and L' a relatively ample divisor on X'. For $\varepsilon \in \mathbb{Q}_{>0}$ small enough, $L_\varepsilon = \pi^*L + \varepsilon L'$ defines an ample $\mathbb{Q}$-divisor on X'. Following the terminology used in differential geometry, we will call the associated $\mathbb{Q}$-polarization L_ε *adiabatic*. The slope of $\mathcal{E}'$ (resp. $(\pi^*\mathcal{F})^{\mathrm{sat}}$) with respect to L_ε admits an expansion in ε, with the coefficient of the smallest exponent given by $\mu_L(\mathcal{E})$ (resp. $\mu_L(\mathcal{F})$). It is straightforward to show that: if $\mathcal{E}$ is stable (resp. unstable) with respect to L, then there is $\varepsilon_0 \in \mathbb{Q}_{>0}$ such that for all $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, $\mathcal{E}'$ is stable (resp. unstable) with respect to L_ε.

The main result of this thesis on fibrations is about the strictly semistable situation. For two coherent sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ on X', we write $\mu_0(\mathcal{F}_1) < \mu_0(\mathcal{F}_2)$ (resp. $\mu_0(\mathcal{F}_1) \leq \mu_0(\mathcal{F}_2)$ or $\mu_0(\mathcal{F}_1) = \mu_0(\mathcal{F}_2)$) when the coefficient of the smallest exponent in the expansion in ε of $\mu_{L_\varepsilon}(\mathcal{F}_2) - \mu_{L_\varepsilon}(\mathcal{F}_1)$ is strictly positive (resp. greater or equal to zero or equal to zero). If $\mathcal{E}$ is a strictly semistable torsion-free sheaf on (X, L), there is a Jordan-Hölder filtration

$$0 = \mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \ldots \subseteq \mathcal{E}_\ell = \mathcal{E}$$

by slope semistable coherent subsheaves with stable quotients of same slope as $\mathcal{E}$. We denote by $\operatorname{Gr}_L(\mathcal{E}) := \bigoplus_{i=1}^{\ell-1} \mathcal{E}_{i+1}/\mathcal{E}_i$ the graded object of $\mathcal{E}$ and $\mathfrak{E}$ the set of equivariant and saturated subsheaves $\mathcal{F} \subseteq \mathcal{E}$ arising in a Jordan-Hölder filtration of $\mathcal{E}$.

Theorem 1.2.8 (Theorem 4.1.9). *Let $\mathcal{E}$ be an equivariant locally free and strictly semistable sheaf on (X, L) such that its graded object is also locally free. Then there is $\varepsilon_0 \in \mathbb{Q}_{>0}$ such that for all $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, the reflexive pullback $\mathcal{E}' = (\pi^*\mathcal{E})^{\vee\vee}$ on (X', L_ε) is:*

1. *stable iff for all $\mathcal{F} \in \mathfrak{E}$, $\mu_0((\pi^*\mathcal{F})^{\vee\vee}) < \mu_0(\mathcal{E}')$,*
2. *strictly semistable iff for all $\mathcal{F} \in \mathfrak{E}$, $\mu_0((\pi^*\mathcal{F})^{\vee\vee}) \leq \mu_0(\mathcal{E}')$,*
3. *unstable iff there is one $\mathcal{F} \in \mathfrak{E}$ with $\mu_0((\pi^*\mathcal{F})^{\vee\vee}) > \mu_0(\mathcal{E}')$.*

Remark 1.2.9. In the case where π is a locally trivial toric fibration, then the assumptions on $\mathcal{E}$ or $\operatorname{Gr}_L(\mathcal{E})$ to be locally free in this theorem are not necessary.

Another case of interest is when the fibration is a blowup. Let X be a smooth projective variety and Z an irreducible subvariety of dimension ℓ with $\ell \leq \dim(X) - 2$. Let $\pi : X' \longrightarrow X$ be the blowup of X along Z and D_0 be its exceptional divisor. For a polarization L on X and $\varepsilon \in \mathbb{Q}_{>0}$ small enough, we consider the polarization L_ε on X' defined by $L_\varepsilon = L - \varepsilon D_0$.

Proposition 1.2.10. *Let Z be a set of invariant points of a smooth toric variety X and $\pi : X' \longrightarrow X$ the blowup along Z. Let $\mathscr{E}$ be an equivariant reflexive sheaf that is strictly semistable on (X, L). Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, the reflexive pullback $\mathscr{E}' := (\pi^*\mathscr{E})^{\vee\vee}$ on (X', L_ε) is:*

1. *strictly semistable iff for any subsheaf $\mathscr{F} \in \mathfrak{E}$, $(\pi^*\mathscr{F})^{\vee\vee}$ is saturated in $\mathscr{E}'$,*
2. *unstable otherwise.*

The result of [9, Proposition 5.1] is more general as it deals with pullbacks of semistable torsion-free sheaves over normal projective varieties, but Proposition 1.2.10 seems to provide more information when $\mathrm{Gr}_L(\mathscr{E})$ is not locally free. If we assume $\ell \geq 1$, for any reflexive sheaf $\mathscr{E}$ on a smooth projective variety X, we show that

$$\mu_{L_\varepsilon}((\pi^*\mathscr{E})^{\vee\vee}) = \mu_L(\mathscr{E}) - \binom{n-1}{\ell-1}\mu_{L_{|Z}}(\mathscr{E}_{|Z})\varepsilon^{n-\ell} + O(\varepsilon^{n-\ell+1}). \tag{1.4}$$

In particular, if X is a smooth toric variety, Z an invariant subvariety of X and $\mathscr{E}$ an equivariant strictly semistable sheaf on (X, L) such that for all $\mathscr{F} \in \mathfrak{E}$, $(\pi^*\mathscr{F})^{\vee\vee}$ is saturated in $\mathscr{E}' := (\pi^*\mathscr{E})^{\vee\vee}$ and

$$\mu_{L_{|Z}}(\mathscr{E}_{|Z}) < \mu_{L_{|Z}}(\mathscr{F}_{|Z}),$$

then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, the pullback $\mathscr{E}'$ is stable on (X', L_ε). In the case where $\ell = 1$, we have the following result.

Theorem 1.2.11 (Theorem 4.2.10)**.** *Let (X, L) be a smooth polarised toric variety. Let $\pi : X' \longrightarrow X$ be the blowup along a T-invariant irreducible curve $Z \subset X$. If $\mathscr{E}$ is an equivariant reflexive sheaf that is strictly semistable on (X, L), then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, the pullback $\mathscr{E}'$ on (X', L_ε) is*

1. *stable iff for all $\mathscr{F} \in \mathfrak{E}$, $(\pi^*\mathscr{F})^{\vee\vee}$ is saturated in $\mathscr{E}'$ and $\dfrac{c_1(\mathscr{E}) \cdot Z}{\operatorname{rk} \mathscr{E}} < \dfrac{c_1(\mathscr{F}) \cdot Z}{\operatorname{rk} \mathscr{F}}$;*
2. *semistable iff for all $\mathscr{F} \in \mathfrak{E}$, $(\pi^*\mathscr{F})^{\vee\vee}$ is saturated in $\mathscr{E}'$ and $\dfrac{c_1(\mathscr{E}) \cdot Z}{\operatorname{rk} \mathscr{E}} \leq \dfrac{c_1(\mathscr{F}) \cdot Z}{\operatorname{rk} \mathscr{F}}$;*
3. *unstable otherwise.*

Remark 1.2.12. As an application of this theorem, in Section 4.2.5 we give an explicit example of strictly semistable sheaf, namely the tangent sheaf of $\mathbb{P}\left(\mathscr{O}_{\mathbb{P}^1}^{\oplus r} \oplus \mathscr{O}_{\mathbb{P}^1}(1)\right)$, that becomes stable or unstable when pulled back to the blowup along a curve.

These results on stability of pullbacks of sheaves along blowups have an application on resolution of singularities. An application of Hironaka's resolution of indeterminacy locus shows that for a given equivariant reflexive sheaf $\mathscr{E}_0 := \mathscr{E}$ on $X_0 := X$, there is a finite sequence of blowups along smooth irreducible torus invariant centers $\pi_i : X_i \longrightarrow X_{i-1}$ for $1 \leq i \leq p$ such that: if we set $\mathscr{E}_i = (\pi_i^*\mathscr{E}_{i-1})^{\vee\vee}$, the sheaf $\mathscr{E}' := \mathscr{E}_p$ is locally free on $X' := X_p$. We denote by $\pi : X_p \longrightarrow X$ the map between X_p and X and $S := X(\mathscr{E})_{\mathrm{sing}}$ the singular locus of $\mathscr{E}$ on X.

Corollary 1.2.13. *Let $\mathscr{E}$ be an equivariant stable sheaf on the polarized toric variety (X, L), there is $\varepsilon_0 \in \mathbb{Q}_{>0}$ such that for all $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, $\mathscr{E}'$ is stable with respect to $\pi^*L - \varepsilon E$ where $E = \pi^{-1}(S)$.*

The number p given above is not explicit. A natural question would be to find an explicit bound on p. In this thesis we describe the singular locus of equivariant reflexive sheaves. From general theory, if X is a smooth complex variety and $\mathscr{E}$ a reflexive sheaf over X, its singular locus $X(\mathscr{E})_{\mathrm{sing}}$ is a Zariski closed subset of X of codimension at least 3. In the toric situation, that is if X is toric and $\mathscr{E}$ is an equivariant sheaf, it is not hard to see that $X(\mathscr{E})_{\mathrm{sing}}$ is a finite union of torus orbit closures. More precisely, if Σ denotes the fan of X, there are $\tau_1, \dots, \tau_r \in \Sigma$, with $\dim(\tau_i) \geq 3$, such that

$$X(\mathscr{E})_{\text{sing}} = \bigcup_{i=1}^{r} V(\tau_i),$$

where $V(\tau_i)$ denotes the closure of the orbit associated to τ_i. We then show that it is possible to prescribe singularities on a sheaf.

Theorem 1.2.14 (Theorem 5.1.6)**.** *Let X be a smooth toric variety with fan Σ. Let $\tau_1, \dots, \tau_m \in \Sigma$ with $\dim(\tau_i) \geq 3$ such that for any $i, j \in \{1, \dots, m\}$ with $i \neq j$, τ_i is not a proper face of τ_j. Then, there exists an equivariant reflexive sheaf $\mathscr{E}$ on X of rank $\sum_{i=1}^{m} \dim(\tau_i) - m$ such that*

$$X(\mathscr{E})_{\text{sing}} = \bigcup_{i=1}^{m} V(\tau_i).$$

Our construction is explicit, and is obtained as a simple generalisation of Hartshorne's example [13, Example 1.9.1].

Organization. The thesis is organized as follows:

- In Chapter 2, we recall the background on toric varieties, equivariant sheaves and stability. The main references of this chapter are [2], [33] and [36].
- In Chapter 3, we study stability of equivariant logarithmic tangent sheaves. We give the proofs of Theorems 1.2.1, 1.2.4, 1.2.5 and also the proofs of Propositions 1.2.2 and 1.2.3. The results of this chapter gave rise to paper [Nap21].
- In Chapter 4 we study the stability of pullback sheaves along fibrations. We prove Theorems 1.2.8 and 1.2.11. We also give the proof of Formula (1.4). The results of this chapter come from [NT22].
- Finally in Chapter 5 we study the singular locus of equivariant reflexive sheaves. We prove Theorem 1.2.14.

1.3. Notations

We give here some notations that are used in the document. An ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ gives an affine variety

$$\mathbb{V}(I) = \{p \in \mathbb{C}^n : f(p) = 0 \text{ for all } f \in I\}$$

and an affine variety $V \subseteq \mathbb{C}^n$ gives the ideal

$$\mathbb{I}(V) = \{f \in \mathbb{C}[x_1, \dots, x_n] : f(p) = 0 \text{ for all } p \in V\}.$$

Let R be a ring and $f \in R$ a nonzero element. We denote by

$$\operatorname{Spm}(R)$$

the maximal spectrum of R and by R_f the localization of R at f.

2

TORIC VARIETIES AND COHERENT SHEAVES

In this chapter, we present different notions that will be discussed in this manuscript: toric varieties, equivariant sheaves and stability of sheaves.

2.1. Toric varieties

In this first section we gather the necessary background about toric varieties [2, Chapter 1, 2, 3].

Definition 2.1.1. A *toric variety* is an irreducible variety X containing a torus $T \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action of T on itself by multiplication extends to an algebraic action of T on X.

Example 2.1.2. The curve $C = \mathbb{V}(x^3 - y^2) \subseteq \mathbb{C}^2$ is an affine toric variety with torus $C \setminus \{0\} = \{(t^2, t^3) : t \in \mathbb{C}^*\}$. As $\mathbb{C}[C]$ is not normal, the variety C is not normal. ♢

2.1.1. Normal toric varieties.

We describe here normal toric varieties. Let N be a rank n lattice and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual lattice with pairing $\langle \cdot, \cdot \rangle : M \times N \longrightarrow \mathbb{Z}$. Then N is the *lattice of one-parameter subgroups* of the n-dimensional complex torus $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$. Note that M is the *character lattice* of T_N.

Notation 2.1.3. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we define $N_{\mathbb{K}} = N \otimes_{\mathbb{Z}} \mathbb{K}$ and $M_{\mathbb{K}} = M \otimes_{\mathbb{Z}} \mathbb{K}$. We denote by $\chi^m : T_N \longrightarrow \mathbb{C}^*$ the character corresponding to $m \in M$ and by $\lambda^u : \mathbb{C}^* \longrightarrow T_N$ the one-parameter subgroup corresponding to $u \in N$.

A *strongly convex polyhedral cone* σ in $N_{\mathbb{R}}$ is a set of the form

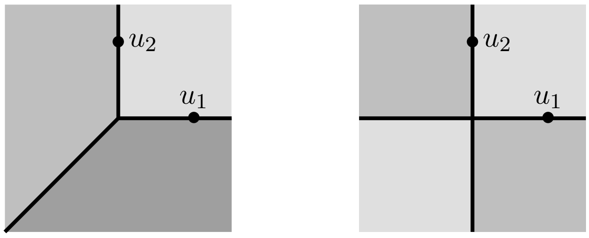
$$\sigma = \mathrm{Cone}(S) = \left\{ \sum_{u \in S} \lambda_u \, u : \lambda_u \geq 0 \right\}$$

where $S \subseteq N_{\mathbb{R}}$ is finite and such that $\sigma \cap (-\sigma) = \{0\}$. Moreover, if $S \subseteq N$, then σ is called *rational*. The dual cone of σ is defined by

$$\sigma^\vee = \{m \in M_{\mathbb{R}} : \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}$$

and a face of σ is given by $\{u \in N_{\mathbb{R}} : \langle m, u \rangle = 0\} \cap \sigma$ for some $m \in \sigma^\vee$.

Definition 2.1.4. A *fan* Σ in $N_{\mathbb{R}}$ is a non-empty set of rational strongly convex polyhedral cones in $N_{\mathbb{R}}$ such that:



(a) Fan of $\mathbb{P}^2$ (b) Fan of $\mathbb{P}^1 \times \mathbb{P}^1$

Figure 2.1: Example of fans in $\mathbb{R}^2$

- Each face of a cone in Σ is also a cone in Σ ;
- The intersection of two cones in Σ is a face of each.

Furthermore, if Σ is a fan, then the *support* of Σ is $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma$.

Notation 2.1.5. Let Σ be a fan. We denote by $\Sigma(r)$ the set of r-dimensional cones of Σ. Elements of $\Sigma(1)$ will be called *rays*. For two cones $\tau, \sigma \in \Sigma$, we write $\tau \preceq \sigma$ if τ is a face of σ and for $\rho \in \Sigma(1)$, we denote by $u_\rho \in N$ its minimal generator.

For $\sigma \in \Sigma$, the lattice of points $S_\sigma = \sigma^\vee \cap M$ is a finitely generated monoid. By [2, Theorem 1.2.18], the variety $U_\sigma = \mathrm{Spm}(\mathbb{C}[S_\sigma])$ is an affine toric variety with torus T_N. We denote by X_Σ the variety obtained by gluing the affine charts $(U_\sigma)_{\sigma\in\Sigma}$ where $U_\sigma \cap U_{\sigma'} = U_{\sigma\cap\sigma'}$ for any $\sigma, \sigma' \in \Sigma$. By [2, Theorem 3.1.5], the variety X_Σ is a normal toric variety with torus T_N. In general, every normal toric variety comes from a fan. This is a consequence of a theorem of Sumihiro.

Theorem 2.1.6 ([35, Theorem 1])**.** *Let the torus T act on a normal variety X. Then every point $p \in X$ has a T-invariant affine open neighborhood.*

Corollary 2.1.7 ([2, Corollary 3.1.8])**.** *Let X be a normal toric variety with torus T and N the lattice of one-parameter subgroups of T. Then, there exists a unique fan Σ in $N_\mathbb{R}$ such that $X \simeq X_\Sigma$.*

From now on, a normal toric variety will be defined by a fan. As we will only consider normal toric varieties, *toric varieties* will mean *normal toric varieties*.

Remark 2.1.8. If Σ_1 is a fan in $(N_1)_\mathbb{R}$ and Σ_2 a fan in $(N_2)_\mathbb{R}$, then the toric variety corresponding to the fan $\Sigma_1 \times \Sigma_2 = \{\sigma_1 \times \sigma_2 : \sigma_i \in (N_i)_\mathbb{R}\}$ in $(N_1 \times N_2)_\mathbb{R}$ is $X_{\Sigma_1} \times X_{\Sigma_2}$.

Example 2.1.9. We assume that $N = \mathbb{Z}$. The only strongly convex cones in $N_\mathbb{R}$ are $\sigma_0 = [0; +\infty[$, $\sigma_1 =]-\infty; 0]$ and $\tau = \{0\}$. The toric variety corresponding to
- $\{\tau\}$ is $\mathbb{C}^*$;
- $\{\tau, \sigma_0\}$ or $\{\tau, \sigma_1\}$ is $\mathbb{C}$;
- $\{\tau, \sigma_0, \sigma_1\}$ is $\mathbb{P}^1$.

This is the list of all 1-dimensional toric varieties viewed as abstract varieties. ♢

Example 2.1.10. Let (u_1, u_2) be the standard basis of $\mathbb{Z}^2$ and Σ_1, Σ_2 be two fans in $\mathbb{R}^2$ such that:

$$\Sigma_1(2) = \{\mathrm{Cone}(u_1, u_2), \mathrm{Cone}(u_2, -(u_1 + u_2)), \mathrm{Cone}(-(u_1 + u_2), u_1)\},$$
$$\Sigma_2(2) = \{\mathrm{Cone}(u_1, u_2), \mathrm{Cone}(u_2, -u_1), \mathrm{Cone}(-u_1, -u_2), \mathrm{Cone}(-u_2, u_1)\}.$$

Figures 2.1a and 2.1b represent fans Σ_1 and Σ_2 respectively. As the affine toric variety corresponding to the cone $\mathrm{Cone}(u_1, u_2)$ is $\mathbb{C}^2$ (as an abstract variety), by gluing the affine charts, we get $X_{\Sigma_1} = \mathbb{P}^2$ and $X_{\Sigma_2} = \mathbb{P}^1 \times \mathbb{P}^1$. ♢

Notation 2.1.11. Let Σ be a fan in $N_{\mathbb{R}}$. For $\sigma \in \Sigma$, we set:

- $\sigma(1) = \Sigma(1) \cap \{\tau \in \Sigma : \tau \preceq \sigma\}$.
- $N_\sigma = \mathrm{Span}(\sigma) \cap N$ and $N(\sigma) = N/N_\sigma$.
- $M(\sigma) = \{m \in M : \langle m, u\rangle = 0 \text{ for all } u \in \sigma\}$ and $M_\sigma = M/M(\sigma)$.
- $T_{N(\sigma)} = \mathrm{Hom}_{\mathbb{Z}}(M(\sigma), \mathbb{C}^*)$ and we denote by γ_σ the point of U_σ such that the torus-orbit $O(\sigma) = T_N \cdot \gamma_\sigma$ is isomorphic to the torus $T_{N(\sigma)}$.

The point $\gamma_\sigma \in U_\sigma$ is called the *distinguished point* of σ.

Theorem 2.1.12 (Orbit-Cone Correspondence, [2, Theorem 3.2.6])**.** *Let X be the toric variety associated to a fan Σ with torus T. Then*

1. *There is a bijective correspondence*

$$\begin{array}{ccc} \{\textit{Cones } \sigma \textit{ in } \Sigma\} & \longleftrightarrow & \{T\textit{-orbits in } X\} \\ \sigma & \longleftrightarrow & O(\sigma) \end{array}$$

   with $\dim O(\sigma) = \dim N_{\mathbb{R}} - \dim \sigma$*.*

2. *The affine open subset U_σ is the union of orbits*

$$U_\sigma = \bigcup_{\tau \preceq \sigma} O(\tau).$$

3. *$\tau \preceq \sigma$ if and only if $O(\sigma) \subseteq \overline{O(\tau)}$, and*

$$\overline{O(\tau)} = \bigcup_{\tau \preceq \sigma} O(\sigma)$$

   where $\overline{O(\tau)}$ denotes the closure in both the classical and Zariski topologies.

For any $\tau \in \Sigma$, we set $V(\tau) = \overline{O(\tau)}$. We have an exact sequence

$$0 \longrightarrow N_\tau \longrightarrow N \longrightarrow N(\tau) \longrightarrow 0.$$

For each cone $\sigma \in \Sigma$ containing τ, let $\overline{\sigma}$ be the image cone in $N(\tau)_{\mathbb{R}}$ under the quotient map $N_{\mathbb{R}} \longrightarrow N(\tau)_{\mathbb{R}}$. Then

$$\mathrm{Star}(\tau) = \{\overline{\sigma} \subseteq N(\tau)_{\mathbb{R}} : \tau \preceq \sigma \in \Sigma\}$$

is a fan in $N(\tau)_{\mathbb{R}}$.

Proposition 2.1.13 ([2, Proposition 3.2.7])**.** *For any $\tau \in \Sigma$, the orbit closure $V(\tau)$ is isomorphic to the toric variety $X_{\mathrm{Star}(\tau)}$.*

If $\rho \in \Sigma(1)$, we denote $V(\rho)$ by D_ρ. For any $\rho \in \Sigma(1)$, D_ρ defines an irreducible T-invariant Weil divisor of X_Σ. Divisors of the form $\sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ are precisely the invariant divisors under the torus action on X_Σ. Thus,

$$\mathrm{WDiv}_T(X_\Sigma) := \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z} D_\rho$$

is the group of invariant Weil divisors on X_Σ. We denote by

- $\mathrm{WDiv}(X_\Sigma)$ the set of Weil divisors on X_Σ ;
- $\mathrm{Div}(X_\Sigma)$ the set of Cartier divisors on X_Σ ;
- $\mathrm{Div}_0(X_\Sigma)$ the set of principal divisors on X_Σ ;
- $\mathrm{Div}_T(X_\Sigma)$ the set of invariant Cartier divisors on X_Σ.

All these sets are in fact additive groups. A particular divisor associated to a variety is its canonical divisor.

Theorem 2.1.14 ([2, Theorem 8.2.3]). *The canonical divisor of a toric variety X_Σ is the torus invariant Weil divisor*

$$K_{X_\Sigma} = -\sum_{\rho\in\Sigma(1)} D_\rho.$$

Two divisors D and D' are *linearly equivalent* on X_Σ, written $D \sim_{\text{lin}} D'$, if $D - D' \in \operatorname{Div}_0(X_\Sigma)$. For Weil and Cartier divisors, linear equivalence classes form the following important groups.

Definition 2.1.15. The *class group* of X_Σ is defined by $\operatorname{Cl}(X_\Sigma) = \operatorname{WDiv}(X_\Sigma)/\operatorname{Div}_0(X_\Sigma)$ and its *Picard group* is $\operatorname{Pic}(X_\Sigma) = \operatorname{Div}(X_\Sigma)/\operatorname{Div}_0(X_\Sigma)$.

By [2, Proposition 4.1.2], for $m \in M$, the character χ^m is a rational function on X_Σ, and its divisor is given by

$$\operatorname{div}(\chi^m) = \sum_{\rho\in\Sigma(1)} \langle m, u_\rho\rangle D_\rho, \tag{2.1}$$

so $\operatorname{div}(\chi^m)$ defines an invariant principal divisor of X_Σ.

Definition 2.1.16. Let X_1, X_2 be normal toric varieties. A morphism $\pi : X_1 \longrightarrow X_2$ is *toric* if π maps the torus T_1 of X_1 into the torus T_2 of X_2 and $\pi_{|T_1} : T_1 \longrightarrow T_2$ is a group homomorphism. We say that π is an equivariant mapping for the T_1- and T_2-actions if for any $t \in T_1, x \in X_1$, $\pi(t \cdot x) = \pi(t) \cdot \pi(x)$.

A normal toric variety X_Σ has a *torus factor* if it is equivariantly isomorphic to the product of a nontrivial torus and a toric variety of smaller dimension. By [2, Proposition 3.3.9], X_Σ has a torus factor if and only if the set $\{u_\rho : \rho \in \Sigma(1)\}$ does not span $N_\mathbb{R}$. If X_Σ has no torus factor, then by [2, Theorem 4.1.3 and 4.2.1] we have the exact sequences

$$0 \longrightarrow M \longrightarrow \operatorname{WDiv}_T(X_\Sigma) \longrightarrow \operatorname{Cl}(X_\Sigma) \longrightarrow 0 \tag{2.2}$$

and

$$0 \longrightarrow M \longrightarrow \operatorname{Div}_T(X_\Sigma) \longrightarrow \operatorname{Pic}(X_\Sigma) \longrightarrow 0. \tag{2.3}$$

Therefore,

Corollary 2.1.17. *If X_Σ has no torus factor, then $|\Sigma(1)| = \dim(X_\Sigma) + \operatorname{rk}\operatorname{Cl}(X)$.*

Let σ be a cone in $N_\mathbb{R}$. We say that σ is *smooth* if its minimal generators form a part of a $\mathbb{Z}$-basis of N and we say that σ is *simplicial* if its minimal generators are linearly independent over $\mathbb{R}$. A fan Σ is *smooth* (resp. *simplicial*) if every cone σ in Σ is smooth (resp. *simplicial*). Finally, we say that Σ is *complete* if $|\Sigma| = N_\mathbb{R}$.

Theorem 2.1.18 ([2, Theorem 3.1.19]). *Let X_Σ be the toric variety defined by the fan Σ. Then :*

1. *X_Σ is a smooth variety if and only if the fan Σ is smooth.*
2. *X_Σ is $\mathbb{Q}$-factorial if and only if the fan Σ is simplicial.*
3. *X_Σ is complete if and only if the fan Σ is complete.*

By [2, Proposition 4.2.6 and 4.2.7], we can characterize smooth and $\mathbb{Q}$-factorial toric varieties by their Picard and class groups.

Proposition 2.1.19. *Let X_Σ be the toric variety defined by the fan Σ. Then:*

1. *X_Σ is smooth if and only if $\operatorname{Pic}(X_\Sigma) = \operatorname{Cl}(X_\Sigma)$.*
2. *X_Σ is $\mathbb{Q}$-factorial if and only if $\operatorname{Pic}(X_\Sigma)$ has finite index in $\operatorname{Cl}(X_\Sigma)$.*

2.1.2. Toric morphisms. Let N_1, N_2 be two lattices with Σ_1 a fan in $(N_1)_\mathbb{R}$ and Σ_2 a fan in $(N_2)_\mathbb{R}$. We denote by X_1 (resp. X_2) the toric variety associated to the fan Σ_1 (resp. Σ_2).

We say that a $\mathbb{Z}$-linear map $\phi : N_1 \longrightarrow N_2$ is *compatible* with the fans Σ_1 and Σ_2 if for every cone $\sigma_1 \in \Sigma_1$, there is a cone $\sigma_2 \in \Sigma_2$ such that $\phi_\mathbb{R}(\sigma_1) \subseteq \sigma_2$ where $\phi_\mathbb{R}$ is the $\mathbb{R}$-linear extension of ϕ. According to [2, Theorem 3.3.4], any toric morphism $\pi : X_1 \longrightarrow X_2$ comes from a $\mathbb{Z}$-linear map $\phi : N_1 \longrightarrow N_2$ that is compatible with Σ_1 and Σ_2.

Proposition 2.1.20 ([2, Proposition 3.3.7]). *Let N_1 be a sublattice of finite index in N_2 and Σ_i a fan in $(N_i)_\mathbb{R}$ such that $\Sigma_1 = \Sigma_2$. We set $G = N_2/N_1$. Then the map $\pi : X_1 \longrightarrow X_2$ induced by the inclusion $N_1 \hookrightarrow N_2$ presents X_2 as the quotient X_1/G.*

The map π in Proposition 2.1.20 is a *geometric quotient* . The following results describe the relation between torus orbits of X_1 and torus orbits of X_2.

Lemma 2.1.21 ([2, Lemma 3.3.21]). *Let $\pi : X_1 \longrightarrow X_2$ be the toric morphism coming from a map $\phi : N_1 \longrightarrow N_2$ that is compatible with Σ_1 and Σ_2. Given $\sigma' \in \Sigma_1$, let $\sigma \in \Sigma_2$ be the minimal cone of Σ_2 containing $\phi_\mathbb{R}(\sigma')$. Then :*

1. *$\pi(\gamma_{\sigma'}) = \gamma_\sigma$ where $\gamma_{\sigma'} \in O(\sigma')$ and $\gamma_\sigma \in O(\sigma)$ are the distinguished points.*
2. *$\pi(O(\sigma')) \subseteq O(\sigma)$ and $\pi(V(\sigma')) \subseteq V(\sigma)$.*

Remark 2.1.22. Note that, if π is surjective, then the inclusions in point 2 of Lemma 2.1.21 are equalities.

A *support function* of Σ is a function $\varphi : |\Sigma| \longrightarrow \mathbb{R}$ that is linear on each cone of Σ. Support functions can be used to caracterize Cartier divisors:

Proposition 2.1.23 ([2, Theorem 4.2.12]). *Let $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ be a Cartier divisor of X_Σ. The support function $\varphi_D : |\Sigma| \longrightarrow \mathbb{R}$ associated to the divisor D satisfies $\varphi_D(u_\rho) = -a_\rho$ for any $\rho \in \Sigma(1)$.*

Following this proposition, it is possible to describe pullback of Cartier divisors by a toric morphism.

Proposition 2.1.24 ([2, Proposition 6.2.7]). *Let D be a torus-invariant Cartier divisor of X_2 with support function $\varphi_D : |\Sigma_2| \longrightarrow \mathbb{R}$. Then there is a unique torus-invariant Cartier divisor $D' := \pi^*D$ on X_1 with the following properties :*

1. *$\mathscr{O}_{X_1}(D') = \pi^*\mathscr{O}_{X_2}(D)$.*
2. *The support function $\varphi_{D'} : |\Sigma_1| \longrightarrow \mathbb{R}$ is given by $\varphi_{D'} = \varphi_D \circ \phi_\mathbb{R}$.*

2.1.3. Toric fibrations. A proper toric morphism $\pi : X_1 \longrightarrow X_2$ is a *fibration* if $\pi_*(\mathscr{O}_{X_1}) = \mathscr{O}_{X_2}$. According to [2, Theorem 3.4.11] and [1, Proposition 2.1], if X_1 and X_2 are complete, then π is a *toric fibration* if and only if the associated map $\phi : N_1 \longrightarrow N_2$ is surjective. We now give specific examples of toric fibrations.

Locally trivial fibrations. Let N, N' be two lattices and $\phi : N' \longrightarrow N$ be a surjective $\mathbb{Z}$-linear map. Let Σ' be a fan in $N'_\mathbb{R}$ and Σ a fan in $N_\mathbb{R}$ compatible with ϕ. We set $N_0 = \mathrm{Ker}(\phi)$ and $\Sigma_0 = \{\sigma \in \Sigma' : \sigma \subseteq (N_0)_\mathbb{R}\}$. We have an exact sequence

$$0 \longrightarrow N_0 \longrightarrow N' \longrightarrow N \longrightarrow 0. \tag{2.4}$$

We say that Σ' is *weakly split by* Σ and Σ_0 if there exists a subfan $\widehat{\Sigma}$ of Σ' such that :

1. $\phi_\mathbb{R}$ maps each cone $\widehat{\sigma} \in \widehat{\Sigma}$ bijectively to a cone $\sigma \in \Sigma$. Furthermore, the map $\widehat{\sigma} \longmapsto \sigma$ define a bijection $\widehat{\Sigma} \longrightarrow \Sigma$.

2. Given $\widehat{\sigma} \in \widehat{\Sigma}$ and $\sigma_0 \in \Sigma_0$, the sum $\widehat{\sigma} + \sigma_0$ lies in Σ', and every cone of Σ' arises in this way.

Moreover, if $\phi(\widehat{\sigma} \cap N') = \sigma \cap N$ for any $\widehat{\sigma} \in \widehat{\Sigma}$ with $\phi_{\mathbb{R}}(\widehat{\sigma}) = \sigma$, we say that Σ' is *split by* Σ and Σ_0.

Theorem 2.1.25 ([2, Theorem 3.3.19]). *If Σ' is split by Σ and Σ_0, then $X_{\Sigma'}$ is a Zariski locally trivial fiber bundle over X_Σ with fiber X_{Σ_0,N_0} where X_{Σ_0,N_0} is the toric variety associated to the fan Σ_0 in $(N_0)_{\mathbb{R}}$.*

In the case where Σ' is weakly split by Σ and Σ_0, for any $\sigma \in \Sigma$ there is a sublattice $N'' \subseteq N$ of finite index such that $\Sigma'(\sigma) = \{\sigma' \in \Sigma' : \phi_{\mathbb{R}}(\sigma') \subseteq \sigma\}$ is split by $\{\tau \in \Sigma : \tau \preceq \sigma\}$ and $\Sigma'(\sigma) \cap \Sigma_0$ when we use the lattice $\phi^{-1}(N'')$ and N''. Let $U_{\sigma,N''}$ be the toric variety associated to the cone σ in $(N'')_{\mathbb{R}}$. There is a commutative diagram

$$\begin{array}{ccc} X_{\Sigma_0,N_0} \times U_{\sigma,N''} & \longrightarrow & X_{\Sigma'} \\ \downarrow & & \downarrow \\ U_{\sigma,N''} & \longrightarrow & U_\sigma \end{array}$$

such that $X_{\Sigma_0,N_0} \times U_{\sigma,N''}$ is the fiber product $X_{\Sigma'} \times_{U_\sigma} U_{\sigma,N''}$.

Corollary 2.1.26. *If Σ' is weakly split by Σ and Σ_0, then the fibers of $\pi : X_{\Sigma'} \longrightarrow X_\Sigma$ are isomorphic to X_{Σ_0,N_0}.*

Blowups. Let Σ be a fan in $N_{\mathbb{R}}$ and assume $\tau \in \Sigma$ with $\dim \tau \geq 2$ has the property that all cones of Σ containing τ are smooth. By Proposition 2.1.13, this is equivalent to the assertion that the orbit closure $V(\tau)$ consists of smooth points of X. Let $u_\tau = \sum_{\rho \in \tau(1)} u_\rho$ and for each cones $\sigma \in \Sigma$ containing τ, set

$$\Sigma^*_\sigma(\tau) = \{\mathrm{Cone}(A) : A \subseteq \{u_\tau\} \cup \sigma(1) \text{ and } \tau(1) \nsubseteq A\}.$$

The *star subdivision* of Σ relative to τ is the fan

$$\Sigma^*(\tau) = \{\sigma \in \Sigma : \tau \nsubseteq \sigma\} \cup \bigcup_{\tau \subseteq \sigma} \Sigma^*_\sigma(\tau).$$

A fan Σ' *refines* Σ if every cone of Σ' is contained in a cone of Σ and $|\Sigma'| = |\Sigma|$. When Σ' refines Σ, the identity mapping $\phi = \mathrm{Id}_N$ is compatible with Σ' and Σ. So there is a toric morphism $\pi : X_{\Sigma^*(\tau)} \longrightarrow X_\Sigma$. Under π, $X_{\Sigma^*(\tau)}$ is the blowup of X_Σ along $V(\tau)$ and the exceptional divisor D_0 of π is the divisor corresponding to the ray $\mathrm{Cone}(u_\tau)$ of $\Sigma^*(\tau)$.

2.1.4. Intersection products.

We first recall some properties of intersection product on varieties over $\mathbb{C}$. We refer to [7, Chapter 1, Section 1.4].

For a variety X and an integer $k \geq 0$, we denote by $A_k(X)$ the k-th Chow group of X. Let $f : X' \longrightarrow X$ be a proper morphism of varieties. For any subvariety V' of X', $V = f(V')$ is a closed subvariety of X. We denote by $R[V]$ (resp. $R[V']$) the residue field of V (resp. of V'). The residue field of V is given by $\mathscr{O}_V/\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of $\mathscr{O}_V$. We set

$$\deg(V'/V) = \begin{cases} [R[V'] : R[V]] & \text{if } \dim V = \dim V' \\ 0 & \text{if } \dim V < \dim V' \end{cases}$$

where $[R[V'] : R[V]]$ denotes the degree of the field extension. Define

$$f_*[V'] = \deg(V'/V)[V].$$

This induces an homomorphism $f_* : A_k(X') \longrightarrow A_k(X)$.

Definition 2.1.27 ([7, Definition 1.4]). Let X be a complete variety over $\mathbb{C}$. If $\alpha = \sum_P n_P[P]$ is a zero-cycle on X, the *degree* of α, denoted $\deg(\alpha)$, is defined by

$$\deg(\alpha) = \sum_P n_P$$

and if α is rationally equivalent to zero, then $\deg(\alpha) = 0$. This gives an homomorphism $\deg : A_0(X) \longrightarrow \mathbb{Z}$. We extend the degree homomorphism to all of A_*X, $\deg : A_*X \longrightarrow \mathbb{Z}$ by defining $\deg(\alpha) = 0$ if $\alpha \in A_k(X)$ with $k > 0$. For any morphism $f : X' \longrightarrow X$ of complete varieties, and any $\alpha' \in A_*X'$,

$$\deg(\alpha') = \deg(f_*\alpha'). \tag{2.5}$$

Proposition 2.1.28 (Projection formula, [7, Proposition 2.3]). *Let D be an effective Cartier divisor on X, $f : X' \longrightarrow X$ a proper morphism, α a k-cycle on X' and g the morphism from $f^{-1}(|D|)\cap|\alpha|$ to $|D| \cap f(|\alpha|)$ induced by f where $|D|$ (resp. $|\alpha|$) is the support of D (resp. α). Then,*

$$g_*(f^*D \cdot \alpha) = D \cdot f_*\alpha \tag{2.6}$$

in $A_{k-1}(|D| \cap f(|\alpha|))$.

We now provide formulas that will be used to compute various intersections of toric divisors. We assume that X is an n-dimensional toric variety given by a complete and simplicial fan Σ. An element $u \in N$ is *primitive* if $\frac{1}{k}u \notin N$ for all $k > 1$. Let $\{u_1, \ldots, u_k\}$ be a set of primitive elements of N such that $\sigma = \mathrm{Cone}(u_1, \ldots, u_k)$ is simplicial. We define $\mathrm{mult}(\sigma)$ as the index of the sublattice $\mathbb{Z}u_1 + \ldots + \mathbb{Z}u_k$ in $N_\sigma = \mathrm{Span}(\sigma) \cap N$.

As Σ is simplicial, according to [8, Section 5.1], one has intersections of cycles or cycle classes only with rational coefficients. The Chow group

$$A^*(X)_\mathbb{Q} = \bigoplus_{p=0}^{n} A^p(X) \otimes \mathbb{Q} = \bigoplus_{p=0}^{n} A_{n-p}(X) \otimes \mathbb{Q}$$

has the structure of graded $\mathbb{Q}$-algebra and,

Proposition 2.1.29. *Let $\tau, \tau', \sigma \in \Sigma$ such that τ and τ' span σ, with $\dim(\sigma) = \dim(\tau) + \dim(\tau')$, then*

$$[V(\tau)] \cdot [V(\tau')] = \frac{\mathrm{mult}(\tau) \cdot \mathrm{mult}(\tau')}{\mathrm{mult}(\sigma)}[V(\sigma)].$$

This proposition is a consequence of the following Lemmas.

Lemma 2.1.30 ([2, Lemma 12.5.1]). *The Chow group $A_k(X)$ is generated by the classes of the orbit closures $V(\sigma)$ of the cones $\sigma \in \Sigma$ of dimension $n - k$.*

Lemma 2.1.31 ([2, Lemma 12.5.2]). *Assume that Σ is complete and simplicial. If $\rho_1, \ldots, \rho_d \in \Sigma(1)$ are distinct, then in $A^\bullet(X)_\mathbb{Q}$, we have*

$$[D_{\rho_1}] \cdot [D_{\rho_2}] \cdots [D_{\rho_d}] = \begin{cases} \dfrac{1}{\mathrm{mult}(\sigma)}[V(\sigma)] & \text{if } \sigma = \rho_1 + \ldots + \rho_d \in \Sigma \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Proposition 2.1.29. Let $\rho_1, \ldots, \rho_q \in \Sigma(1)$ distinct such that $\tau = \rho_1 + \ldots + \rho_p$ and $\tau' = \rho_{p+1} + \ldots \rho_q$ with $p < q$. By Lemma 2.1.31, we get

$$\frac{1}{\mathrm{mult}(\sigma)}[V(\sigma)] = ([D_{\rho_1}] \cdots [D_{\rho_p}]) \cdot ([D_{\rho_{p+1}}] \cdots [D_{\rho_q}]) = \frac{1}{\mathrm{mult}(\tau) \cdot \mathrm{mult}(\tau')}[V(\tau)] \cdot [V(\tau')].$$

□

For any $m \in M$, we have $\operatorname{div}(\chi^m) = 0$ in $A_{n-1}(X)$. In the case where $\rho \in \Sigma(1)$ is a ray of $\sigma \in \Sigma$, there is $m \in M$ such that $V(\sigma)$ is not contained in the support of $D_\rho + \operatorname{div}(\chi^m)$. We then set

$$[D_\rho] \cdot [V(\sigma)] = [D_\rho + \operatorname{div}(\chi^m)] \cdot [V(\sigma)]. \tag{2.7}$$

2.1.5. Polytopes and ample divisors of complete toric varieties. A polytope P in $M_\mathbb{R}$ is the convex hull of a finite set $S \subseteq M_\mathbb{R}$, i.e

$$P = \operatorname{Conv}(S) = \left\{ \sum_{u \in S} \lambda_u \, u : \lambda_u \geq 0 \text{ and } \sum_{u \in S} \lambda_u = 1 \right\}.$$

Moreover, if S is a subset of M, we say that P is a *lattice polytope*. A subset $Q \subseteq P$ is a face of P, written $Q \preceq P$, if there are $u \in N_\mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$ such that

$$Q = \{m \in M_\mathbb{R} : \langle m, u \rangle = b\} \cap P \quad \text{and} \quad P \subseteq \{m \in M_\mathbb{R} : \langle m, u \rangle \geq b\}.$$

When $\dim P = \dim M_\mathbb{R}$, the polytope P has a nice presentation with its facets (faces of P of codimension one):

$$P = \{m \in M_\mathbb{R} : \langle m, u_F \rangle \geq -a_F \text{ for all facets } F \preceq P\} \tag{2.8}$$

where $u_F \in N_\mathbb{R}$ is an *inward-pointing facet normal* of the facet F. For any face Q of P, there is a cone σ_Q of $N_\mathbb{R}$ defined by

$$\sigma_Q = \operatorname{Cone}(u_F : F \text{ contains } Q). \tag{2.9}$$

Thus, for a facet $F \preceq P$, σ_F is the ray generated by u_F and $\sigma_P = \{0\}$ since $\{0\}$ is the cone generated by the empty set. Hence, the set

$$\Sigma_P = \{\sigma_Q : Q \preceq P\}.$$

is a complete fan in $N_\mathbb{R}$. We denote by X_P the toric variety associated to Σ_P and we define the divisor D_P associated to P by

$$D_P = \sum_F a_F D_F$$

where D_F is the divisor of X_P corresponding to the ray $\operatorname{Cone}(u_F)$.

Let $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ be a Cartier divisor on a complete toric variety X_Σ. The polyhedron

$$P_D = \{m \in M_\mathbb{R} : \langle m, u_\rho \rangle \geq -a_\rho \text{ for all } \rho \in \Sigma(1)\}$$

is a polytope and

$$\Gamma\left(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)\right) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m.$$

Definition 2.1.32. Let $\mathscr{L}$ be the sheaf of sections of the rank one vector bundle $\pi : \mathscr{V} \longrightarrow X$ on a variety X. A subspace $W \subseteq \Gamma(X, \mathscr{L})$ *has no basepoints* if for every $p \in X$, there is $s \in W$ with $s(p) \neq 0$.

Definition 2.1.33. Let D be a Cartier divisor on a complete toric variety X_Σ. We set $W = \Gamma\left(X_\Sigma, \mathscr{O}_{X_\Sigma}(D)\right)$.

1. The divisor D and the line bundle $\mathscr{O}_{X_\Sigma}(D)$ are *very ample* when W has no basepoints and the map $\phi_D : X \longrightarrow \mathbb{P}(W^\vee)$ is a closed embedding.

2. The divisor D and the line bundle $\mathscr{O}_{X_\Sigma}(D)$ are *ample* when kD is very ample for some integer $k > 0$.

On toric varieties, we can characterize ample divisors by using the Toric Kleiman Criterion.

Theorem 2.1.34 ([2, Theorem 6.3.13]). *Let D be a Cartier divisor on a complete toric variety X_Σ. Then D is ample if and only if $D \cdot C > 0$ for all torus-invariant irreducible curves $C \subseteq X_\Sigma$.*

If D is an invariant ample divisor on X_Σ, then D (or $\mathscr{O}_{X_\Sigma}(D)$) will be called a *polarization*. We call the pair (X_Σ, D) or $(X_\Sigma, \mathscr{O}_{X_\Sigma}(D))$ a *polarized variety*. We have the following result between polytopes and polarized varieties.

Theorem 2.1.35 ([2, Theorem 6.2.1]). *The maps $P \longmapsto (X_P, D_P)$ and $(X_\Sigma, D) \longmapsto P_D$ define bijections between the sets*

$$\{P \subseteq M_\mathbb{R} : P \text{ is a full dimensional lattice polytope}\}$$

and

$$\{(X_\Sigma, D) : \Sigma \text{ a complete fan in } N_\mathbb{R},\ D \text{ a torus-invariant ample divisor on } X_\Sigma\}.$$

Moreover, these maps are inverses of each other.

According to Theorem 2.1.35, the polarized toric variety (X_Σ, D) gives a polytope $P \subseteq M_\mathbb{R}$. For each $\rho \in \Sigma(1)$ we denote by P^ρ the facet of P corresponding to the ray $\rho \in \Sigma(1)$. We recall that a lattice M defines a measure ν on $\mathrm{M}_\mathbb{R}$ as the pullback of the Haar measure on $\mathrm{M}_\mathbb{R}/\mathrm{M}$. It is determined by the properties

i. ν is translation invariant,

ii. $\nu(\mathrm{M}_\mathbb{R}/\mathrm{M}) = 1$.

For all $\rho \in \Sigma(1)$, we denote by $\mathrm{vol}(P^\rho)$ the volume of P^ρ with respect to the measure determined by the affine span of $P^\rho \cap M$. Danilov in [3, Section 11] shows that:

Proposition 2.1.36 ([3, §11.12]). *Let (X_Σ, D) be a polarized toric variety corresponding to a lattice polytope P. For all ray $\rho \in \Sigma(1)$, $\mathrm{vol}(P^\rho) = D_\rho \cdot D^{n-1}$.*

Remark 2.1.37. We will use this proposition to compute the slope of sheaves.

2.2. Examples of toric varieties of Picard rank one and two

2.2.1. Toric varieties of Picard rank one. Let $q_0, q_1, \dots, q_n \in \mathbb{N}^*$ such that

$$\gcd(q_0, \dots, q_n) = 1.$$

We set $N = \mathbb{Z}^{n+1}/\mathbb{Z} \cdot (q_0, \dots, q_n)$. The dual lattice of N is

$$M = \{(a_0, \dots, a_n) \in \mathbb{Z}^{n+1} : a_0\, q_0 + \dots + a_n\, q_n = 0\}.$$

We denote by $\{u_i : 0 \leq i \leq n\}$ the images in N of the standard basis vectors in $\mathbb{Z}^{n+1}$. So the relation $q_0\, u_0 + q_1\, u_1 + \dots + q_n\, u_n = 0$ holds in N. The toric variety associated to the simplicial fan $\Sigma = \{\mathrm{Cone}(A) : A \subsetneq \{u_0, \dots, u_n\}\}$ is the weighted projective space $\mathbb{P}(q_0, q_1, \dots, q_n)$.

Proposition 2.2.1 ([20, Corollary 2.3]). *Let Σ be a complete simplicial fan in $N_\mathbb{R}$ such that $|\Sigma(1)| = n+1$, where $n = \dim X_\Sigma$. There is a weighted projective space $\mathbb{P}(q_0, \dots, q_n)$ and a finite abelian group H acting on $\mathbb{P}(q_0, \dots, q_n)$ such that $X_\Sigma \simeq \mathbb{P}(q_0, \dots, q_n)/H$. Moreover $X_\Sigma \simeq \mathbb{P}^n$ when Σ is smooth.*

Proof. Let $\{u_i : 0 \leq i \leq n\}$ be the set of ray generators of Σ. There are $q_i \in \mathbb{Z}$ such that $\sum_i q_i u_i = 0$. As Σ is complete, we deduce that for any $i \in \{0, \dots, n\}$, $q_i > 0$. Hence, we can assume that $\gcd(q_0, \dots, q_n) = 1$. Let N' be the sublattice of N generated by $u_0, \dots, u_n$. The toric variety associated to Σ in $N'_{\mathbb{R}}$ is $\mathbb{P}(q_0, \dots, q_n)$. By Proposition 2.1.20, we get $X_\Sigma \simeq \mathbb{P}(q_0, \dots, q_n)/H$ where $H = N/N'$. □

Let $q_0, \dots, q_n \in \mathbb{N}^*$ such that $1 = q_0 \leq q_1 \leq \dots \leq q_n$ and $q_i | \sum_{j=0}^n q_j$ for $i \in \{0, \dots, n\}$. We denote by $(e_1, \dots, e_n)$ the dual basis of $(u_1, \dots, u_n)$. Let $X = \mathbb{P}(q_0, \dots, q_n)$ and $L = -K_X$. The $\mathbb{Q}$-Cartier divisor L is ample on X. We denote by P the polytope corresponding to (X, L). The point $m = m_1 e_1 + \dots + m_n e_n \in M$ lies in P if and only if $m_i \geq -1$ for $1 \leq i \leq n$ and

$$q_1 m_1 + \dots + q_n m_n \leq q_0 \,.$$

Hence $(-1, \dots, -1)$ is a vertex of P. The others vertices $(m_1, \dots, m_n)$ of P are given by $m_k = -1$ for $k \in \{1, \dots, n\} \setminus \{i\}$ and $q_1 m_1 + \dots + q_n m_n = q_0$; thus

$$m_i = \frac{q_0 + q_1 + \dots + q_n}{q_i} - 1.$$

It follows that

$$P = \mathrm{Conv}(0, k_1 e_1, \dots, k_n e_n) - (1, \dots, 1)$$

where $q_i\, k_i = \sum_{j=0}^n q_j$.

2.2.2. Smooth toric varieties of Picard rank two. Let X be a smooth toric variety of dimension n with fan Σ in $\mathbb{R}^n$ such that $\mathrm{rk}\,\mathrm{Pic}(X) = 2$. By [2, Theorem 7.3.7] due to Kleinschmidt [22], there are $r, s \in \mathbb{N}^*$ with $r + s = n$ and $a_1, \dots, a_r \in \mathbb{N}$ with $a_1 \leq a_2 \leq \dots \leq a_r$ such that

$$X = \mathbb{P}\left(\mathscr{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^r \mathscr{O}_{\mathbb{P}^s}(a_i)\right). \tag{2.10}$$

We denote by $\pi : X \longrightarrow \mathbb{P}^s$ the projection to the base $\mathbb{P}^s$. By [2, Section 7.3], the rays of Σ are given by the half-lines generated by $w_0, w_1, \dots, w_s, v_0, v_1, \dots, v_r$ where $(w_1, \dots, w_s)$ is the standard basis of $\mathbb{Z}^s \times 0_{\mathbb{Z}^r}$, $(v_1, \dots, v_r)$ the standard basis of $0_{\mathbb{Z}^s} \times \mathbb{Z}^r$,

$$v_0 = -(v_1 + \dots + v_r) \quad \text{and} \quad w_0 = a_1 v_1 + \dots + a_r v_r - (w_1 + \dots + w_s).$$

The maximal cones of Σ are given by

$$\mathrm{Cone}(w_0, \dots, \widehat{w_j}, \dots, w_s) + \mathrm{Cone}(v_0, \dots, \widehat{v_i}, \dots, v_r)$$

where $j \in \{0, \dots, s\}$ and $i \in \{0, \dots, r\}$. We denote by D_{v_i} the divisor corresponding to the ray $\mathrm{Cone}(v_i)$ and D_{w_j} the divisor corresponding to the ray $\mathrm{Cone}(w_j)$. We have the following linear equivalence,

$$\begin{cases} D_{v_i} \sim_{\mathrm{lin}} D_{v_0} - a_i D_{w_0} & \text{for } i \in \{1, \dots, r\} \\ D_{w_j} \sim_{\mathrm{lin}} D_{w_0} & \text{for } j \in \{1, \dots, s\} \end{cases}. \tag{2.11}$$

By (2.11), we deduce that $\mathrm{Pic}(X)$ is generated by D_{v_0} and D_{w_0}.

Proposition 2.2.2 ([4, Proposition 4.2.1])**.** *Let $D = \alpha D_{w_0} + \beta D_{v_0}$ be an invariant divisor of X with $\alpha, \beta \in \mathbb{Z}$. Then, the divisor D is ample if and only if $\alpha > 0$ and $\beta > 0$.*

By Theorem 2.1.14, the anti-canonical divisor of X is given by

$$-K_X = \sum_{i=0}^{r} D_{v_i} + \sum_{j=0}^{s} D_{w_j} \sim_{\mathrm{lin}} (s+1-a_1-\ldots-a_r)D_{w_0} + (r+1)D_{v_0}. \tag{2.12}$$

Thus, X is a Fano variety if and only if $a_1+\ldots+a_r \leq s$.

Remark 2.2.3. The sheaf $\mathscr{O}_X(D_{v_0})$ of X is isomorphic to the twisting sheaf of Serre $\mathscr{O}_X(1)$. Therefore, for any $\alpha, \beta \in \mathbb{N}^*$, $\mathscr{O}_X(\alpha D_{w_0} + \beta D_{v_0}) \cong \pi^*\mathscr{O}_{\mathbb{P}^s}(\alpha) \otimes \mathscr{O}_X(\beta)$.

2.2.3. Polytope of a polarized toric variety of Picard rank two. Let X be a smooth toric variety given by (2.10) and $L = \pi^*\mathscr{O}_{\mathbb{P}^s}(\nu)\otimes\mathscr{O}_X(1)$ a $\mathbb{Q}$-divisor of X with $\nu \in \mathbb{Q}_{>0}$. For $k \in \{1,\ldots,s\}$, we set $\Delta_k = \mathrm{Conv}(0, w_1, \ldots, w_k)$. By [14, Section 4], the polytope corresponding to the $\mathbb{Q}$-polarized variety (X, L) is given by

$$P = \mathrm{Conv}\left(\nu\Delta_s \times \{0\} \cup (a_1+\nu)\Delta_s \times \{v_1\} \cup \ldots \cup (a_r+\nu)\Delta_s \times \{v_r\}\right) .$$

We denote by P^{v_i} (resp. P^{w_j}) the facet of P corresponding to the ray $\mathrm{Cone}(v_i)$ (resp. $\mathrm{Cone}(w_j)$). The facet P^{v_i} is the convex hull of

$$\nu\Delta_s \times \{0\} \cup \ldots \cup (a_{i-1}+\nu)\Delta_s \times \{v_{i-1}\} \cup (a_{i+1}+\nu)\Delta_s \times \{v_{i+1}\} \cup \ldots \cup (a_r+\nu)\Delta_s \times \{v_r\}$$

and P^{w_i} is isomorphic to

$$\nu\Delta_{s-1} \times \{0\} \cup (a_1+\nu)\Delta_{s-1} \times \{v_1\} \cup \ldots \cup (a_r+\nu)\Delta_{s-1} \times \{v_r\} .$$

Proposition 2.2.4 ([14, Proposition 4.3]). *Let $c_0, c_1, \ldots, c_r \in \mathbb{N}$ and $\nu \in \mathbb{Q}_{>0}$. The volume of the polytope*

$$P = \mathrm{Conv}\left((c_0+\nu)\Delta_s \times \{0\} \cup (c_1+\nu)\Delta_s \times \{v_1\} \cup \ldots \cup (c_r+\nu)\Delta_s \times \{v_r\}\right)$$

is given by

$$\sum_{k=0}^{s} \binom{s+r}{k} \left(\sum_{d_0+\ldots+d_r=s-k} c_0^{d_0}\cdots c_r^{d_r} \right) \nu^k.$$

Therefore, for any $j \in \{0,\ldots,s\}$,

$$\mathrm{vol}(P^{w_j}) = \sum_{k=0}^{s-1} \binom{s+r-1}{k} \left(\sum_{d_1+\ldots+d_r=s-k-1} a_1^{d_1}\cdots a_r^{d_r} \right) \nu^k$$

and

$$\mathrm{vol}(P^{v_0}) = \sum_{k=0}^{s} \binom{s+r-1}{k} \left(\sum_{d_1+\ldots+d_r=s-k} a_1^{d_1}\cdots a_r^{d_r} \right) \nu^k .$$

If $i \in \{1,\ldots,r\}$, we have

$$\mathrm{vol}(P^{v_i}) = \sum_{k=0}^{s} \binom{s+r-1}{k} \left(\sum_{\substack{d_1+\ldots+d_{i-1} \\ +d_{i+1}+\ldots+d_r=s-k}} a_1^{d_1}\cdots a_{i-1}^{d_{i-1}}a_{i+1}^{d_{i+1}}\cdots a_r^{d_r} \right) \nu^k.$$

All these formula will be used in Chapter 3 when we study the stability of the logarithmic tangent sheaves on toric varieties of Picard rank two.

2.3. Stability of equivariant reflexive sheaves

2.3.1. Coherent and equivariant sheaves. Let $X = \operatorname{Spm}(R)$ be an affine variety. A nonzero element $f \in R$ gives the localization R_f such that $X_f = \operatorname{Spm}(R_f)$ is the open subset $X \setminus \mathbb{V}(f)$. Given an R-module M, we get the R_f-module $M_f = M \otimes_R R_f$. According to [12, Proposition II.5.1], there is a unique sheaf $\widetilde{M}$ of $\mathscr{O}_X$-modules such that

$$\widetilde{M}(X_f) = M_f$$

for every nonzero $f \in R$.

Definition 2.3.1. Let X be a locally Noetherian scheme and $\mathscr{F}$ a sheaf of $\mathscr{O}_X$-modules. We say that $\mathscr{F}$ is *quasicoherent* if X has an affine open cover $\{U_\alpha\}$, $U_\alpha = \operatorname{Spm}(R_\alpha)$, such that for each α, there is an R_α-module M_α satisfying $\mathscr{F}_{|U_\alpha} = \widetilde{M}_\alpha$. Moreover, if each M_α is a finetely generated R_α-module, then we say that $\mathscr{F}$ is *coherent*.

Definition 2.3.2. Let X be a locally Noetherian scheme and $\mathscr{E}$ a coherent sheaf. A coherent subsheaf $\mathscr{F}$ of $\mathscr{E}$ is *saturated* if the quotient sheaf $\mathscr{E}/\mathscr{F}$ is torsion-free. Given a point $x \in X$, the fiber of $\mathscr{E}$ at x is defined as a vector space

$$\mathscr{E}(x) = \mathscr{E}_x \otimes_{\mathscr{O}_{X,x}} R(x) \tag{2.13}$$

where $R(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ with $\mathfrak{m}_x$ the maximal ideal of $\mathscr{O}_{X,x}$.

Let X be a normal toric variety with torus T. We denote by $\theta : T \times X \longrightarrow X$ the action of T on X, $\mu : T \times T \longrightarrow T$ the group multiplication, $\operatorname{pr}_2 : T \times X \longrightarrow X$ the projection onto the second factor and $\operatorname{pr}_{23} : T \times T \times X \longrightarrow T \times X$ the projection onto the second and the third factor.

Definition 2.3.3. A coherent sheaf $\mathscr{E}$ on X is T*-equivariant* if it is equipped with an isomorphism $\Phi : \theta^*\mathscr{E} \longrightarrow \operatorname{pr}_2^*\mathscr{E}$ such that

$$(\mu \times \operatorname{Id}_X)^*\Phi = \operatorname{pr}_{23}^*\Phi \circ (\operatorname{Id}_T \times \theta)^*\Phi. \tag{2.14}$$

We call Φ a T-linearization of $\mathscr{E}$. A morphism of equivariant coherent sheaves is a morphism compatible with the linearizations.

Remark 2.3.4. If G is an algebraic group acting on the affine toric variety $Y = \operatorname{Spm}(R)$, we define an action of G on R by setting : for any $g \in G$ and $\varphi \in R$, $g \cdot \varphi = \left(\phi_{g^{-1}}\right)^* \varphi$ where $\phi_g(x) = g \cdot x$.

For $t \in T$, let $\alpha_t : X \longrightarrow T \times X$ and $\phi_t : X \longrightarrow X$ be the morphisms given by $\alpha_t(x) = (t, x)$ and $\phi_t(x) = \theta(t, x)$. If $\mathscr{E}$ is an equivariant sheaf on X with linearization Φ, then

$$\Phi_t := \alpha_t^*\Phi : \phi_t^*\mathscr{E} \xrightarrow{\cong} \mathscr{E}$$

is an isomorphism such that, for any $t, t' \in T$, the cocycle condition (2.14) factors as follows:

$$\begin{array}{ccc} (\phi_{t'\cdot t})^*\mathscr{E} & \xrightarrow{\quad\Phi_{t'\cdot t}\quad} & \mathscr{E} \\ & {\scriptstyle \phi_t^*\Phi_{t'}}\searrow \qquad \nearrow{\scriptstyle \Phi_t} & \\ & \phi_t^*\mathscr{E} & \end{array} \tag{2.15}$$

Let Σ be the fan of X, $\mathscr{E}$ an equivariant coherent sheaf on X and $\sigma \in \Sigma$. We denote the space $\Gamma(U_\sigma, \mathscr{E})$ by E^σ. For $s \in E^\sigma$ and $g \in T$, we denote by $\phi_g^* s \in E^\sigma$ the canonically lifted section of $\phi_g^* \mathscr{E}$. We define the action of T on E^σ by setting : for $g \in T$ and $s \in E^\sigma$,

$$g \cdot s = \Phi_{g^{-1}} \left((\phi_{g^{-1}})^* s \right) \ .$$

As the sheaf $\mathscr{E}$ is coherent, there is a decomposition $E^\sigma = \bigoplus_{m \in M} E_m^\sigma$ such that for any $g \in T$ and $e \in E_m^\sigma$, $g \cdot e = \chi^{-m}(g) e$. This decomposition makes E^σ an M-graded $\mathbb{C}[S_\sigma]$-module.

2.3.2. Families of filtrations of equivariant reflexive sheaves. For $\sigma \in \Sigma$, we define an order relation $\preceq_\sigma$ on M by setting $m \preceq_\sigma m'$ if and only if $m' - m \in S_\sigma$. We write $m \prec_\sigma m'$ if we have $m \preceq_\sigma m'$ but not $m' \preceq_\sigma m$.

Definition 2.3.5 ([33, Definition 5.17])**.** Let E be a finite dimensional vector space and let for each $\sigma \in \Sigma$ a set of vector subspaces $\{E_m^\sigma\}_{m \in M}$ of E. We say that this family is a *multifiltration* if:

1. For $\sigma \in \Sigma$ and $m \preceq_\sigma m'$, E_m^σ is contained in $E_{m'}^\sigma$. Moreover $E = \sum_{m \in M} E_m^\sigma$.
2. For each chain $\cdots \prec_\sigma m_{i-1} \prec_\sigma m_i \prec_\sigma \cdots$ of elements of M, there exists $i_0 \in \mathbb{Z}$ such that $E_{m_i}^\sigma = 0$ for all $i \leq i_0$.
3. For each $\sigma \in \Sigma$, there exist only finitely many vector spaces E_m^σ such that
$$E_m^\sigma \not\subseteq \sum_{m' \prec_\sigma m} E_{m'}^\sigma \ .$$
4. (compatibility condition) For each $\tau \preceq \sigma$ with $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-m_\tau)$ we consider with respect to the preorder $\preceq_\sigma$ the ascending chains $m + i \cdot m_\tau$ for $i \geq 0$. By condition 3 and because E is finite dimensional, the sequence of vector subspaces $E_{m + i \cdot m_\tau}$ becomes stationary for some $i_m^\tau \in \mathbb{Z}$. We require that $E_m^\tau = E_{m + i_m^\tau \cdot m_\tau}$ for all $m \in M$.

Remark 2.3.6. Note that we are using increasing filtrations here, rather than decreasing as in [23].

Let $\widehat{E} = \{E_m^\sigma\}_m$ be a family of vector spaces. For each relation $m \preceq_\sigma m'$, let there be given a vector space homomorphism $\chi_{m,m'}^\sigma : E_m^\sigma \longrightarrow E_{m'}^\sigma$ such that $\chi_{m,m}^\sigma = \mathrm{Id}$ and $\chi_{m,m''}^\sigma = \chi_{m',m''}^\sigma \circ \chi_{m,m'}^\sigma$ for each triple $m \preceq_\sigma m' \preceq_\sigma m''$. If $\widehat{F} = \{F_m^\sigma\}_m$ is another family of vector spaces with vector space homomorphisms $\psi_{m,m'}^\sigma$, then a morphism $\widehat{\phi} : \widehat{E} \longrightarrow \widehat{F}$ is a set of vector space homomorphisms $\{\phi_m^\sigma : E_m^\sigma \longrightarrow F_m^\sigma\}_{m \in M}$ such that $\phi_{m'}^\sigma \circ \chi_{m,m'}^\sigma = \psi_{m,m'}^\sigma \circ \phi_m^\sigma$ for all $m, m' \in M$ with $m \preceq_\sigma m'$. We then have the following result:

Theorem 2.3.7 ([33, Theorem 5.18])**.** *The category of torsion free equivariant coherent sheaves is equivalent to the category of families of multifiltrations of finite dimensional vector spaces.*

We recall that a *reflexive sheaf* on X is a coherent sheaf $\mathscr{E}$ that is canonically isomorphic to its double dual $\mathscr{E}^{\vee\vee}$. So for an equivariant reflexive sheaf $\mathscr{E}$ on X and $\sigma \in \Sigma$, we have

$$\Gamma(U_\sigma, \mathscr{E}) = \Gamma\left(\bigcup_{\rho \in \sigma(1)} U_\rho, \mathscr{E} \right) = \bigcap_{\rho \in \sigma(1)} \Gamma(U_\rho, \mathscr{E}) \ .$$

Hence, an equivariant reflexive sheaf $\mathscr{E}$ on a toric variety X_Σ is uniquely determined by the family of multifiltrations $(\{E_m^\rho\})_{\rho \in \Sigma(1)}$ of a vector space E.

Remark 2.3.8. The vector space E can be seen as the fiber $\mathscr{E}(x_0)$ where x_0 is the identity element of T and we define the vector subspaces $\{E^\rho_m\}$ as follows: let $\gamma_\rho \in O(\rho)$ be the distinguished point, we set

$$E^\rho_m = \left\{ e \in E : \lim_{t\cdot x_0 \to \gamma_\rho, t\in T} \chi^m(t)(t\cdot e) \text{ exists} \right\}$$

where $t \cdot e$ is an element of $\mathscr{E}(t\cdot x_0)$.

Let $\rho \in \Sigma(1)$. As $E^\rho_m = E^\rho_{m'}$ if $m - m' \in M(\rho) = \rho^\perp \cap M$ and $M/M(\rho) \cong \mathbb{Z}$, we set

$$E^\rho(\langle m, u_\rho\rangle) = E^\rho_m.$$

To a family of multifiltrations

$$\mathbb{E} := \left(E, \{E^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\right) \tag{2.16}$$

with $E^\rho(j) \subseteq E^\rho(j+1)$, we can assign an equivariant reflexive sheaf $\mathscr{E} := \mathfrak{K}(\mathbb{E})$ defined by

$$\Gamma(U_\sigma, \mathscr{E}) := \bigoplus_{m\in M} \bigcap_{\rho\in\sigma(1)} E^\rho(\langle m, u_\rho\rangle) \otimes \chi^m \tag{2.17}$$

for all positive dimensional cones $\sigma \in \Sigma$, while $\Gamma(U_{\{0\}}, \mathscr{E}) = E \otimes \mathbb{C}[M]$. From now on, the family of multifiltrations given in (2.16) will be called a *family of filtrations*.

Notation 2.3.9. For any $\sigma \in \Sigma$, we write

$$E^\sigma = \Gamma(U_\sigma, \mathscr{E}) = \bigoplus_{m\in M} E^\sigma_m \otimes \chi^m$$

where E^σ_m is a vector space.

Example 2.3.10 (Filtrations of invertible sheaves)**.** Let X be a toric variety associated to a complete fan Σ and $D = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$ be an invariant Weil divisor of X. For any $\rho \in \Sigma(1)$,

$$\Gamma(U_\rho, \mathscr{O}_X(D)) = \Gamma(U_\rho, \mathscr{O}_{U_\rho}(a_\rho D_\rho)) \cong \bigoplus_{m\in M,\, \langle m,u_\rho\rangle \geq -a_\rho} \mathbb{C}\cdot\chi^m.$$

Therefore,

$$E^\rho(j) = \begin{cases} 0 & \text{if } j < -a_\rho \\ \mathbb{C} & \text{if } j \geq -a_\rho \end{cases}$$

for $\rho \in \Sigma(1)$ and $j \in \mathbb{Z}$ is the family of filtrations of $\mathscr{O}_X(D)$. ♢

Example 2.3.11 (Tangent sheaf)**.** The family of filtrations of the tangent sheaf $\mathscr{T}_X$ of X is given by

$$E^\rho(j) = \begin{cases} 0 & \text{if } j < -1 \\ \operatorname{Span}(u_\rho) & \text{if } j = -1 \\ N \otimes_\mathbb{Z} \mathbb{C} & \text{if } j > -1 \end{cases}.$$

This is a consequence of Theorem 3.1.5. ♢

We now describe equivariant locally free sheaves on toric varieties. By [19, Theorem 3.5], equivariant locally-free sheaves over affine toric varieties are free. The local freeness property is given by Klyachko's compatibility condition for the filtrations $E^\rho(i)$ in [23, Theorem 2.2.1]. Here we give this condition in term of increasing filtrations.

Proposition 2.3.12 ([32, Proposition 4.24])**.** *The sheaf $\mathscr{E}$ is locally free if and only if for any $\sigma \in \Sigma$ there exists a multiset $A_\sigma \subseteq M/M(\sigma)$ of size $\operatorname{rk}(\mathscr{E})$ and a T-eigenspace decomposition $E = \bigoplus_{m\in A_\sigma} \mathbb{E}^\sigma_m$ such that*

$$E^\rho(i) = \bigoplus_{m\in A_\sigma,\, \langle m,u_\rho\rangle \leq i} \mathbb{E}^\sigma_m \tag{2.18}$$

for any $\rho \in \sigma(1)$.

2.3.3. Some stability notions. In this part, we are interested in the notion of slope stability. We refer to the paper of Takemoto [36] for the definitions. We denote by $\mathrm{Amp}(X) \subseteq N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ the ample cone of X. Let $\mathscr{E}$ be a torsion-free coherent sheaf on X. The *degree* of $\mathscr{E}$ with respect to an ample class $L \in \mathrm{Amp}(X)$ is the real number obtained by intersection

$$\deg_L(\mathscr{E}) = c_1(\mathscr{E}) \cdot L^{n-1}$$

and its *slope* with respect to L is given by

$$\mu_L(\mathscr{E}) = \frac{\deg_L(\mathscr{E})}{\mathrm{rk}(\mathscr{E})}.$$

Definition 2.3.13. A torsion-free coherent sheaf $\mathscr{E}$ is said to be *slope semistable* (or *semistable* for short) with respect to $L \in \mathrm{Amp}(X)$ if for any proper coherent subsheaf of lower rank $\mathscr{F}$ of $\mathscr{E}$, one has

$$\mu_L(\mathscr{F}) \leq \mu_L(\mathscr{E}).$$

When strict inequality always holds, we say that $\mathscr{E}$ is *stable*. We say that $\mathscr{E}$ is *polystable* if it is the direct sum of stable subsheaves of the same slope. Finally $\mathscr{E}$ is said to be *unstable* with respect to L if $\mathscr{E}$ is not semistable with respect to L.

Notation 2.3.14. Let $\mathscr{E}$ be a torsion-free coherent sheaf on X. We denote by

$$\begin{aligned} \mathrm{Stab}(\mathscr{E}) &= \{L \in \mathrm{Amp}(X) : \mathscr{E} \text{ is stable with respect to } L\} \quad \text{and} \\ \mathrm{sStab}(\mathscr{E}) &= \{L \in \mathrm{Amp}(X) : \mathscr{E} \text{ is semistable with respect to } L\}. \end{aligned}$$

Proposition 2.3.15 ([25, Claim 2 of Proposition 4.13])**.** *A reflexive polystable sheaf on X is a semistable sheaf on X isomorphic to a (finite, nontrivial) direct sum of reflexive stable sheaves. Let $\mathscr{E}$ be a semistable reflexive sheaf on X. Then $\mathscr{E}$ contains a unique maximal reflexive polystable subsheaf of the same slope as $\mathscr{E}$.*

If $\mathscr{E}$ is an equivariant reflexive sheaf on a normal toric variety X, according to [25, Proposition 4.13] and [14, Proposition 2.3], it is enough to test slope inequalities for equivariant and reflexive saturated subsheaves.

Proposition 2.3.16. *Let $\mathscr{E}$ be an equivariant reflexive sheaf on X. Then $\mathscr{E}$ is semistable (resp. stable) with respect to L if and only if for all saturated equivariant reflexive subsheaves $\mathscr{F}$ of $\mathscr{E}$, $\mu_L(\mathscr{F}) \leq \mu_L(\mathscr{E})$ (resp. $\mu_L(\mathscr{F}) < \mu_L(\mathscr{E})$).*

Let $\mathscr{E}$ be an equivariant reflexive sheaf on a normal toric variety X given by the family of filtrations $\left(E, \{E^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}}\right)$. From the previous proposition, it is crucial for us to understand the description of equivariant reflexive and saturated subsheaves of $\mathscr{E}$ in terms of families of filtrations. This is the content of the following lemma.

Lemma 2.3.17 ([5])**.** *Let F be a vector subspace of E and $\mathscr{F}$ an equivariant reflexive subsheaf of $\mathscr{E}$ given by the family of filtrations $(F, \{F^\rho(i)\})$ with $F^\rho(i) \subseteq E^\rho(i)$. Then $\mathscr{F}$ is saturated in $\mathscr{E}$ if and only if for all $\rho \in \Sigma(1), i \in \mathbb{Z}$,*

$$F^\rho(i) = E^\rho(i) \cap F.$$

Proof. As $\mathscr{F}$ is an equivariant sheaf, the quotient sheaf $\mathscr{E}/\mathscr{F}$ is equivariant. For any $\rho \in \Sigma(1)$, one has $\Gamma(U_\rho, \mathscr{E}/\mathscr{F}) = \Gamma(U_\rho, \mathscr{E})/\Gamma(U_\rho, \mathscr{F})$ and

$$\Gamma(U_\rho, \mathscr{E}/\mathscr{F})_m = E^\rho(\langle m, u_\rho\rangle)/F^\rho(\langle m, u_\rho\rangle) \otimes \chi^m$$

for any $m \in M$.

We assume that there is $(\rho,\, i) \in \Sigma(1) \times \mathbb{Z}$ such that $F^\rho(i) \neq E^\rho(i) \cap F$. Let

$$i_0 = \min\{i \in \mathbb{Z} : F^\rho(i) \neq E^\rho(i) \cap F\}$$

and $v_\rho \in M$ such that $M_\mathbb{R} = \mathbb{R}v_\rho \oplus \mathrm{Span}(u_\rho)^\perp$ with $\langle v_\rho, u_\rho\rangle = 1$. There is $e \in F \cap (E^\rho(i_0) \setminus F^\rho(i_0))$ such that $0 \neq \overline{e} \in E^\rho(i_0)/F^\rho(i_0)$ where $\overline{e}$ denotes the image of e in $E^\rho(i_0)/F^\rho(i_0)$. Then, as $e \in F$ and as $F^\rho(i) = F$ for i large enough, there is $m \in S_\rho$ with $\langle m + i_0 v_\rho, u_\rho\rangle = i$, such that

$$e \otimes \chi^{m+i_0 v_\rho} \in F^\rho(\langle m + i_0\, v_\rho, u_\rho\rangle) \otimes \chi^{m+i_0 v_\rho} = F \otimes \chi^{m+i_0 v_\rho}.$$

Thus $\overline{e} \otimes \chi^{m+i_0 v_\rho} = 0$ in $\Gamma(U_\rho, \mathscr{E}/\mathscr{F})$. Hence, $\mathscr{E}/\mathscr{F}$ has nonzero torsion. Therefore, if $\mathscr{E}/\mathscr{F}$ is torsion-free, then $F^\rho(i) = E^\rho(i) \cap F$ for any $(\rho, i) \in \Sigma(1) \times \mathbb{Z}$.

We now assume that for any $(\rho, i) \in \Sigma(1) \times \mathbb{Z}$, $F^\rho(i) = E^\rho(i) \cap F$. We use Notation 2.3.9. For any $\sigma \in \Sigma$, we set

$$E^\sigma = \Gamma(U_\sigma, \mathscr{E}) = \bigoplus_{m \in M} E^\sigma_m \otimes \chi^m \quad \text{and} \quad F^\sigma = \Gamma(U_\sigma, \mathscr{F}) = \bigoplus_{m \in M} F^\sigma_m \otimes \chi^m.$$

We will show that E^σ/F^σ is a torsion-free $\mathbb{C}[S_\sigma]$-module. Let $e \in E^\sigma_m$ and $\overline{e}$ be its image in E^σ_m/F^σ_m such that there is $m' \in S_\sigma$ with

$$\overline{e} \otimes \chi^{m+m'} = 0$$

in $\Gamma(U_\sigma, \mathscr{E})_{m+m'}/\Gamma(U_\sigma, \mathscr{F})_{m+m'}$. We have

$$e \otimes \chi^{m+m'} \in \Gamma(U_\sigma, \mathscr{F})_{m+m'}$$

and then $e \in F^\sigma_{m+m'} \subseteq F$. As $F^\sigma_m = E^\sigma_m \cap F$, we get $e \in F^\sigma_m$. Hence, E^σ/F^σ is a torsion-free $\mathbb{C}[S_\sigma]$-module. Therefore, $\mathscr{E}/\mathscr{F}$ is torsion-free. □

Notation 2.3.18. Let F be a vector subspace of E. We denote by $\mathscr{E}_F$ the saturated subsheaf of $\mathscr{E}$ defined by the family of filtrations $(F,\, \{F^\rho(j)\})$ where $F^\rho(j) = F \cap E^\rho(j)$.

By [25, Corollary 3.18], the first Chern class of an equivariant reflexive sheaf $\mathscr{E}$ with family of filtrations $(E, \{E^\rho(j)\})$ is given by

$$c_1(\mathscr{E}) = -\sum_{\rho \in \Sigma(1)} \iota_\rho(\mathscr{E})\, D_\rho \tag{2.19}$$

where

$$\iota_\rho(\mathscr{E}) = \sum_{j \in \mathbb{Z}} j\, (\dim(E^\rho(j)) - \dim(E^\rho(j-1))).$$

Therefore, for any $L \in \mathrm{Amp}(X)$,

$$\mu_L(\mathscr{E}) = -\frac{1}{\mathrm{rk}(\mathscr{E})} \sum_{\rho \in \Sigma(1)} \iota_\rho(\mathscr{E}) \deg_L(D_\rho). \tag{2.20}$$

According to these two formulas and Example 2.3.10, if $\mathscr{E}$ is the invertible sheaf $\mathscr{O}_X(D_\rho)$, then for any $L \in \mathrm{Amp}(X)$,

$$\mu_L(\mathscr{E}) = \deg_L(D_\rho). \tag{2.21}$$

Thanks to Lemma 2.3.17, if $\mathscr{E}$ is an equivariant reflexive sheaf, we have the following control on the number of values used in comparing slopes:

Lemma 2.3.19. *The set* $\{\mu_L(\mathscr{E}_F) : F \subseteq E$ *with* $0 < \dim F < \dim E\}$ *is finite.*

Proof. For any $\rho \in \Sigma(1)$, there is $(j_\rho, J_\rho) \in \mathbb{Z}^2$ such that $E^\rho(j) = \{0\}$ if $j < j_\rho$ and $E^\rho(j) = E$ if $j \geq J_\rho$. For any vector subspace F of E, we have

$$\iota_\rho(\mathscr{E}_F) = \sum_{j=j_\rho}^{J_\rho} j\,(\dim(E^\rho(j) \cap F) - \dim(E^\rho(j-1) \cap F)).$$

As the set $\{\dim(E^\rho(j) \cap F) - \dim(E^\rho(j-1) \cap F) : F \subsetneq E,\ j \in \mathbb{Z}\}$ is finite, we deduce that $\{\iota_\rho(\mathscr{E}_F) : F \subsetneq E\}$ is finite. We can conclude using Formula (2.20). □

3

STABILITY OF EQUIVARIANT LOGARITHMIC TANGENT SHEAVES

In this chapter, we study slope-stability of the equivariant logarithmic tangent sheaf $\mathscr{T}_X(-\log D)$ where X is a normal toric variety and D a reduced invariant Weil divisor of X. In the first part, we give a condition on a divisor D that ensures the existence of a polarization L such that $\mathscr{T}_X(-\log D)$ is (semi)stable. In the other sections, we give a complete description of divisors D and polarizations L such that $\mathscr{T}_X(-\log D)$ is (semi)stable with respect to L when X has Picard rank one or two.

3.1. Description of equivariant logarithmic tangent sheaves

3.1.1. Logarithmic tangent sheaves. We recall here the definition of the logarithmic tangent sheaf of a log pair (X, D) where X is a normal projective variety of dimension n and D a reduced Weil divisor on X.

Definition 3.1.1. We say that a pair (X, D) is *log-smooth* if X is smooth and D is a reduced snc (simple normal crossing) divisor. We denote by $(X, D)_{\mathrm{reg}}$ the snc locus of the pair (X, D), that is, the locus of points $x \in X$ where (X, D) is log-smooth in a neighborhood of x.

If (X, D) is log-smooth, we define the logarithmic tangent bundle $T_X(-\log D)$ as the dual of the bundle of logarithmic differential form $\Omega^1_X(\log D)$ where $\Omega^1_X(\log D)$ is defined in [17, §1]. By [21, Definition 4] and [34, §1], we can see the space of sections of $T_X(-\log D)$ as the set of vector fields on X which are tangent to D at its smooth points.

Let $(z_1, \ldots, z_n)$ be a local coordinate system for X. If D is given by $(z_1 \cdots z_k = 0)$, then $T_X(-\log D)$ as a sheaf is the locally free $\mathscr{O}_X$-module generated by

$$z_1 \frac{\partial}{\partial z_1}, \ldots, z_k \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_{k+1}}, \ldots, \frac{\partial}{\partial z_n}.$$

Definition 3.1.2 ([11, Definition 3.4])**.** Let (X, D) be a log pair and $X_0 = (X, D)_{\mathrm{reg}}$. The logarithmic tangent sheaf of (X, D), denoted by $\mathscr{T}_X(-\log D)$, is defined as $j_* T_{X_0}(-\log D_{|X_0})$ where $j : X_0 \longrightarrow X$ is the open immersion.

The sheaf $\mathscr{T}_X(-\log D)$ (as well as its dual) is coherent; by [13, Proposition 1.6], this sheaf is reflexive. We now consider the case where X is a toric variety with torus T. Let Σ be the fan of X and let X_0 be the toric variety corresponding to the fan $\Sigma^1 = \Sigma(0) \cup \Sigma(1)$. We denote by $j : X_0 \longrightarrow X$ the open immersion.

Proposition 3.1.3. *Let D be a reduced Weil divisor on X. The sheaf $\Omega^1_X(\log D)$ is equivariant compatibly with its subsheaf Ω^1_X if and only if D is a torus invariant divisor of X.*

Proof. We assume that D is an invariant divisor under the torus action. Let D_0 be the restriction of D on X_0. For $t \in T$, let $\phi_t : X \longrightarrow X$ be the map defined by $\phi_t(x) = t \cdot x$ and Φ_t the map defined by $\Phi_t = (d\phi_t)^{-1}$ where $d\phi_t$ is the differential of ϕ_t. If $\mathscr{E} = TX_0$, we have an isomorphism $\Phi_t : \phi_t^*\mathscr{E} \longrightarrow \mathscr{E}$ and the diagram (2.15) is verified. Now if we replace $\mathscr{E}$ by $T_{X_0}(-\log D_0)$, the diagram (2.15) remains true; so $T_{X_0}(-\log D_0)$ is an equivariant subsheaf of T_{X_0}. Hence $\Omega^1_{X_0}(\log D_0)$ is an equivariant sheaf compatibly with its subsheaf $\Omega^1_{X_0}$. As

$$\Omega^1_X(\log D) \cong j_*\Omega^1_{X_0}(\log D_0)\ , \tag{3.1}$$

we deduce that $\Omega^1_X(\log D)$ is an equivariant sheaf compatibly with its subsheaf Ω^1_X.

We now assume that $\Omega^1_X(\log D)$ is an equivariant sheaf compatibly with its subsheaf Ω^1_X. We write $D = \sum_{k=1}^{s} D_k$ where the D_k are irreducible Weil divisors of X.

First case. We assume that X is smooth. By [6, Properties 2.3] we have an exact sequence of equivariant sheaves

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log D) \longrightarrow \bigoplus_{k=1}^{s} \mathscr{O}_{D_k} \longrightarrow 0$$

where $\mathscr{O}_{D_k}$ is viewing as a sheaf on X via extension by zero. Hence, for any $x \in X$, the sequence

$$0 \longrightarrow \Omega^1_{X,x} \longrightarrow \Omega^1_X(\log D)_x \longrightarrow \bigoplus_{k=1}^{s} \mathscr{O}_{D_k,x} \longrightarrow 0$$

is exact. Let $Z = X \setminus D$. If there are $x \in Z$ and $t \in T$ such that $t \cdot x \in D$, then

$$\bigoplus_{k=1}^{s} \mathscr{O}_{D_k,x} \cong \bigoplus_{k=1}^{s} \mathscr{O}_{D_k,t\cdot x}\ ;$$

this is absurd. Thus, for any $t \in T$, $t \cdot Z \subseteq Z$, that is $t \cdot Z = Z$. Therefore, for any $t \in T$, $t \cdot D = D$; thus, D is a torus invariant divisor.

Second case. We assume that X is a normal variety. By (3.1), as $\Omega^1_X(\log D)$ is equivariant, we also have the same property for $\Omega^1_{X_0}(\log D_0)$. By the first case, D_0 is an invariant divisor under the action of T on X_0. As $\operatorname{codim}(X \setminus X_0) \geq 2$, we deduce that D is the Zariski closure of D_0 on X. Thus, D is an invariant divisor under the action of T on X. □

Remark 3.1.4. For the first part of the converse, another proof consists in observing that the determinant of $\Omega^1_X(\log D)$ is $\mathscr{O}_X(K_X + D)$, while the determinant of Ω^1_X is $\mathscr{O}_X(K_X)$. If $\Omega^1_X(\log D)$ is equivariant compatibly with its subsheaf Ω^1_X, then $\mathscr{O}_X$ is an equivariant subsheaf of $\mathscr{O}_X(D)$. This means that D is a torus invariant divisor of X.

3.1.2. Families of filtrations of logarithmic tangent sheaves. Let X be a toric variety of dimension n associated to the fan Σ. By Proposition 3.1.3, the logarithmic tangent sheaf $\mathscr{T}_X(-\log D)$ is an equivariant subsheaf of the tangent sheaf if and only if

$$D = \sum_{\rho \in \Delta} D_\rho$$

where $\Delta \subseteq \Sigma(1)$. For $\rho \in \Sigma(1)$, we set $E^\rho = \Gamma(U_\rho, \mathscr{T}_X(-\log D))$.

Theorem 3.1.5. *Let $\Delta \subseteq \Sigma(1)$ and $D = \sum_{\rho\in\Delta} D_\rho$ be an invariant reduced divisor of X. The family of filtrations $\left(E, \{E^\rho(j)\}_{\rho\in\Sigma(1), j\in\mathbb{Z}}\right)$ corresponding to the logarithmic tangent sheaf $\mathscr{T}_X(-\log D)$ is given by*

$$E^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ N_\mathbb{C} & \text{if } j \geq 0 \end{cases} \qquad \text{if } \rho \in \Delta$$

and by

$$E^\rho(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ \operatorname{Span}(u_\rho) & \text{if } j = -1 \\ N_\mathbb{C} & \text{if } j \geq 0 \end{cases} \qquad \text{if } \rho \notin \Delta\,.$$

Proof. We first assume that X is smooth. By [21, Proposition 1], the following sequence is exact.

$$0 \longrightarrow T_X(-\log D) \longrightarrow T_X \longrightarrow \bigoplus_{\rho\in\Delta} \mathscr{O}_{D_\rho}(D_\rho) \longrightarrow 0 \tag{3.2}$$

By the orbit-cone correspondence (cf. Theorem 2.1.12), if $\rho \in \Delta$, then $U_\rho \cap D = U_\rho \cap D_\rho$, otherwise $U_\rho \cap D = \varnothing$. Therefore, for any $\rho \notin \Delta$, $\Gamma(U_\rho, T_X(-\log D)) \cong \Gamma(U_\rho, T_X)$. According to (3.2) and Theorem 2.3.7, we can reduce the problem to the case where Δ contains one ray. For the rest of the proof, we assume that $\Delta = \{\rho_0\}$. Let $\rho \in \Sigma(1)$ and $(u_1, \dots, u_n)$ be a basis of N such that $u_1 = u_\rho$. We denote by $(e_1, \dots, e_n)$ the dual basis of $(u_1, \dots, u_n)$ and we set $x_i = \chi^{e_i}$. We have $\mathbb{C}[S_\rho] = \mathbb{C}[x_1, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$.

First case : We assume that $\rho = \rho_0$. As on U_ρ the divisor D is defined by the equation $x_1 = 0$, we have

$$E^\rho = \left(\mathbb{C}[S_\rho] \cdot x_1 \frac{\partial}{\partial x_1}\right) \oplus \left(\bigoplus_{i=2}^{n} \mathbb{C}[S_\rho] \cdot \frac{\partial}{\partial x_i}\right).$$

We set

$$L_1^\rho = \bigoplus_{m\in S_\rho} \mathbb{C} \cdot \chi^{m+e_1} \frac{\partial}{\partial x_1} \quad \text{and for } i \in \{2, \dots, n\}, \quad L_i^\rho = \bigoplus_{m\in S_\rho} \mathbb{C} \cdot \chi^m \frac{\partial}{\partial x_i}\,.$$

According to Remark 2.3.4, for any $t \in T$ and $m \in M$, $t \cdot \chi^m = \chi^{-m}(t)\chi^m$, so $t \cdot dx_i = \chi^{-e_i}(t) dx_i$. Thus, we have $t \cdot \frac{\partial}{\partial x_i} = \chi^{e_i}(t) \frac{\partial}{\partial x_i}$. For $i \in \{1, \dots, n\}$, we write

$$L_i^\rho = \bigoplus_{m\in M} (L_i^\rho)_m \quad \text{where} \quad (L_i^\rho)_m = \{f \in L_i^\rho : t \cdot f = \chi^{-m}(t) f\}\,.$$

We have

$$(L_1^\rho)_m = \begin{cases} \mathbb{C} \cdot \chi^{m+e_1} \dfrac{\partial}{\partial x_1} & \text{if } 0 \preceq_\rho m \\ 0 & \text{otherwise} \end{cases}$$

and for $i \in \{2, \dots, n\}$,

$$(L_i^\rho)_m = \begin{cases} \mathbb{C} \cdot \chi^{m+e_i} \dfrac{\partial}{\partial x_i} & \text{if } -e_i \preceq_\rho m \\ 0 & \text{otherwise} \end{cases}.$$

In local coordinates $(x_1, \dots, x_n)$, the tangent space of the Lie group T at the identity element is generated by $\left(\frac{\partial}{\partial x_i}\right)_{1\leq i\leq n}$. As the tangent space of T at the identity element is isomorphic to $N_\mathbb{C}$, for all $i \in \{1, \dots, n\}$, we can identify $\frac{\partial}{\partial x_i}$ with u_i. For $i \in \{1, \dots, n\}$, we set $\mathbb{L}_i^\rho = \operatorname{Span}(u_i)$. Let $m \in M$.

- If $i = 1$ and $0 \preceq_\rho m$, then $(L_i^\rho)_m$ is isomorphic to with $\mathbb{L}_1^\rho \otimes \chi^m$.
- If $i \geq 2$ and $-e_i \preceq_\rho m$, then $(L_i^\rho)_m$ is isomorphic to $\mathbb{L}_i^\rho \otimes \chi^m$.

We set $j = \langle m, u_1 \rangle$. The condition $0 \preceq_\rho m$ is equivalent to $j \geq 0$ and for $i \in \{2, \ldots, n\}$, $-e_i \preceq_\rho m$ is equivalent to $j \geq 0$. Thus, for any $i \in \{1, \ldots, n\}$, we set

$$L_i^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ \mathbb{L}_i^\rho & \text{if } j \geq 0 \end{cases} .$$

By construction, $\{L_i^\rho(j)\}$ is the family of filtrations of L_i^ρ. As $E^\rho = \bigoplus_{m \in M} E^\rho(\langle m, u_1 \rangle) \otimes \chi^m$ where $E^\rho(\langle m, u_1 \rangle) \cong \bigoplus_{i=1}^n L_i^\rho(\langle m, u_\rho \rangle)$, we get

$$E^\rho(j) \cong \begin{cases} 0 & \text{if } j \leq -1 \\ N_{\mathbb{C}} & \text{if } j \geq 0 \end{cases} .$$

Second case : We assume that $\rho \neq \rho_0$. As $U_\rho \cap D = \varnothing$, we have

$$E^\rho = \bigoplus_{i=1}^n \mathbb{C}[S_\rho] \cdot \frac{\partial}{\partial x_i} = \bigoplus_{i=1}^n \left(\bigoplus_{m \in S_\rho} \mathbb{C} \cdot \chi^m \frac{\partial}{\partial x_i} \right) .$$

For all $i \in \{1, \ldots, n\}$, we set $L_i^\rho = \mathbb{C}[S_\rho] \cdot \dfrac{\partial}{\partial x_i}$. We have

$$L_i^\rho = \bigoplus_{m \in M} (L_i^\rho)_m \quad \text{where} \quad (L_i^\rho)_m = \begin{cases} \mathbb{C} \cdot \chi^{m+e_i} \dfrac{\partial}{\partial x_i} & \text{if } -e_i \preceq_\rho m \\ 0 & \text{oherwise} \end{cases} .$$

For $m \in M$, we set $j = \langle m, u_1 \rangle$. The condition $-e_i \preceq_\rho m$ is equivalent to $j \geq -\langle e_i, u_1 \rangle$. Thus, for all $i \in \{2, \ldots, n\}$, the filtrations of L_i^ρ are given by

$$L_i^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ \mathbb{L}_i^\rho & \text{if } j \geq 0 \end{cases}$$

and the filtrations of L_1^ρ are given by

$$L_1^\rho(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ \mathbb{L}_i^\rho & \text{if } j \geq -1 \end{cases} .$$

As in the first case, we get

$$E^\rho(j) \cong \begin{cases} 0 & \text{if } j \leq -2 \\ \operatorname{Span}(u_\rho) & \text{if } j = -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } j \geq 0 \end{cases} .$$

If X is normal, then for any $\rho \in \Sigma(1)$, $\Gamma(U_\rho, \mathscr{T}_X(-\log D)) \cong \Gamma(U_\rho, T_{X_0}(-\log D_{|X_0}))$ where X_0 is the toric variety of the fan $\Sigma(0) \cup \Sigma(1)$. By using the smooth case, we get the proof. □

The sheaf of regular sections of the trivial vector bundle $X \times \mathbb{C} \longrightarrow X$ of rank 1 is $\mathscr{O}_X$. By Example 2.3.10 the family of filtrations $\big(F, \{F^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}}\big)$ of $\mathscr{O}_X$ is given by

$$F^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ \mathbb{C} & \text{if } j \geq 0 \end{cases} .$$

Corollary 3.1.6. *Let* $D = \sum_{\rho \in \Sigma(1)} D_\rho$. *Then the morphism* $\mathscr{O}_X \otimes \operatorname{Lie}(T) \longrightarrow \mathscr{T}_X(-\log D)$ *is an isomorphism.*

Proof. The family of filtrations $(E, \{E^\rho(j)\})$ of $\mathscr{E} = \mathscr{T}_X(-\log D)$ is given by

$$E^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ N \otimes_{\mathbb{Z}} \mathbb{C} & \text{if } j \geq 0 \end{cases}.$$

Hence, for any $\rho \in \Sigma(1)$, $\Gamma(U_\rho, \mathscr{E}) \cong \mathscr{O}_X(U_\rho) \otimes N_{\mathbb{C}}$. ☐

Notation 3.1.7. Let G be a vector subspace of $N_{\mathbb{C}}$. We denote by $\mathscr{E}_G$ the subsheaf of $\mathscr{E} = \mathscr{T}_X(-\log D)$ defined by the family of filtrations $\big(E_G, \{G^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$ where $E_G = G$ and $G^\rho(j) = E^\rho(j) \cap G$. If $\rho \in \Delta$ or $u_\rho \notin G$, then

$$G^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ G & \text{if } j \geq 0 \end{cases}.$$

If $\rho \notin \Delta$ and $u_\rho \in G$, then

$$G^\rho(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ \operatorname{Span}(u_\rho) & \text{if } j = -1 \\ G & \text{if } j \geq 0 \end{cases}.$$

3.1.3. Decomposition of equivariant logarithmic tangent sheaves. In this part, we give some conditions on Σ and Δ which ensure that the logarithmic tangent sheaf is decomposable. We first recall the family of filtrations of a direct sum of equivariant reflexive sheaves.

Proposition 3.1.8 ([18, Section 6.3]). *Let $\mathscr{F}$ and $\mathscr{G}$ be two equivariant reflexive sheaves with $\big(F, \{F^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$ and $\big(G, \{G^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big)$ for families of filtrations. The family of filtrations of $\mathscr{F} \oplus \mathscr{G}$ is given by*

$$\big(F \oplus G, \{(F\oplus G)^\rho(j)\}_{\rho\in\Sigma(1),\, j\in\mathbb{Z}}\big) \quad \text{where} \quad (F\oplus G)^\rho(j) = F^\rho(j) \oplus G^\rho(j). \tag{3.3}$$

We assume that X is a toric variety without torus factor. We denote by p the rank of the class group $\operatorname{Cl}(X)$ of X. By Corollary 2.1.17, we have $\operatorname{card}(\Sigma(1)) = n + p$.

Proposition 3.1.9. *Let $\Delta \subsetneq \Sigma(1)$ such that $|\Delta| = p$ and $D = \sum_{\rho\in\Delta} D_\rho$. If $N_{\mathbb{R}} = \operatorname{Span}(u_\rho : \rho \in \Sigma(1) \setminus \Delta)$, then $\mathscr{E} = \mathscr{T}_X(-\log D)$ is decomposable and*

$$\mathscr{E} = \bigoplus_{\rho\in\Sigma(1)\setminus\Delta} \mathscr{O}_X(D_\rho).$$

Proof. We set $\Sigma(1) \setminus \Delta = \{\rho_1, \ldots, \rho_n\}$. According to Example 2.3.10, for all $k \in \{1, \ldots, n\}$, the family of filtrations $(F_k, \{F_k^\rho(j)\})$ of $\mathscr{O}_X(D_{\rho_k})$ is given by

$$F_k^\rho(j) = \begin{cases} 0 & \text{if } j < 0 \\ \operatorname{Span}(u_\rho) & \text{if } j \geq 0 \end{cases} \quad \text{if } \rho \neq \rho_k$$

and

$$F_k^\rho(j) = \begin{cases} 0 & \text{if } j < -1 \\ \operatorname{Span}(u_\rho) & \text{if } j \geq -1 \end{cases} \quad \text{if } \rho = \rho_k.$$

For all $\rho \in \Sigma(1)$ and $j \in \mathbb{Z}$, we have

$$\bigoplus_{k=1}^{n} F_k^\rho(j) = \begin{cases} 0 & \text{if } j \leq -1 \\ N_{\mathbb{C}} & \text{if } j \geq 0 \end{cases} \quad \text{if } \rho \in \Delta$$

and

$$\bigoplus_{k=1}^{n} F_k^{\rho}(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ \mathrm{Span}(u_\rho) & \text{if } j = -1 \\ N_{\mathbb{C}} & \text{if } j \geq 0 \end{cases} \quad \text{if } \rho \notin \Delta \ .$$

Hence, by (3.3) and Theorem 3.1.5 we get $\mathscr{E} = \bigoplus_{k=1}^{n} \mathscr{O}_X(D_{\rho_k})$. □

We also have the following result, the proof is similar to the proof of Proposition 3.1.9.

Proposition 3.1.10. *We assume that* Δ *satisfies* $1 + p \leq \mathrm{card}(\Delta) \leq n + p - 1$. *The sheaf* $\mathscr{E} = \mathscr{T}_X(-\log D)$ *is decomposable and* $\mathscr{E} = \mathscr{E}_G \oplus \mathscr{E}_F$ *where* $G = \mathrm{Span}(u_\rho : \rho \in \Sigma(1) \setminus \Delta)$ *and* F *a vector subspace of* $N_{\mathbb{C}}$ *such that* $N_{\mathbb{C}} = G \oplus F$.

3.1.4. An instability condition for logarithmic tangent sheaves. Let $\Delta \subsetneq \Sigma(1)$ and

$$D = \sum_{\rho \in \Delta} D_\rho$$

be an invariant reduced Weil divisor on X. Let $\left(E_G, \{G^\rho(j)\}_{\rho \in \Sigma(1), j \in \mathbb{Z}}\right)$ be the family of filtrations corresponding to the subsheaf $\mathscr{E}_G$ (cf. Notation 2.3.18) of $\mathscr{E} = \mathscr{T}_X(-\log D)$ where $G \subseteq N_{\mathbb{C}}$ is a vector subspace. By Equation (2.20), if L is a polarization of X, we have

$$\mu_L(\mathscr{E}) = \frac{1}{n} \sum_{\rho \notin \Delta} \deg_L(D_\rho) \tag{3.4}$$

and

$$\mu_L(\mathscr{E}_G) = \frac{1}{\dim G} \sum_{\rho \notin \Delta \text{ and } u_\rho \in G} \deg_L(D_\rho) \ . \tag{3.5}$$

Therefore,

$$\mu_L(\mathscr{E}) - \mu_L(\mathscr{E}_G) = \left(\frac{1}{n} - \frac{1}{\dim G}\right) \sum_{\rho \notin \Delta,\, u_\rho \in G} \deg_L(D_\rho) + \frac{1}{n} \sum_{\rho \notin \Delta,\, u_\rho \notin G} \deg_L(D_\rho). \tag{3.6}$$

According to Proposition 2.3.16, we have the following Lemma.

Lemma 3.1.11. *To check the stability of* $\mathscr{E}$ *with respect to* L*, it suffices to compare* $\mu_L(\mathscr{E})$ *with* $\mu_L(\mathscr{E}_F)$ *when* $F \subseteq \mathrm{Span}(u_\rho : \rho \notin \Delta)$ *and* $1 \leq \dim F \leq n - 1$.

Proof. Let G be a vector subspace of E such that $1 \leq \dim G \leq n - 1$. We set

$$F = \mathrm{Span}(u_\rho : \rho \notin \Delta \text{ and } u_\rho \in G).$$

If $\dim F \neq 0$, then by (3.5), we get $\mu_L(\mathscr{E}_G) \leq \mu_L(\mathscr{E}_F)$. □

Proposition 3.1.12. *If* $1 \leq \mathrm{card}(\Sigma(1) \setminus \Delta) \leq n - 1$, *then for any* $L \in \mathrm{Amp}(X)$, *the logarithmic tangent sheaf* $\mathscr{E} = \mathscr{T}_X(-\log D)$ *is unstable with respect to* L.

Proof. We assume that $\Sigma(1) \setminus \Delta = \{\rho_1, \ldots, \rho_k\}$ where $1 \leq k \leq n - 1$ and we denote by D_j the divisor corresponding to $\rho_j = \mathrm{Cone}(u_j)$. For $G = \mathrm{Span}(u_1, \ldots, u_k)$, we have

$$\mu_L(\mathscr{E}) - \mu_L(\mathscr{E}_G) = \left(\frac{1}{n} - \frac{1}{\dim G}\right) \sum_{j=1}^{k} \deg_L(D_j) < 0$$

because the numbers $\deg_L(D_j)$ are positive. Thus, $\mathscr{E}$ is unstable with respect to L. □

Corollary 3.1.13. *We set* $p = \operatorname{rk}\operatorname{Cl}(X)$. *If* $1 + p \leq \operatorname{card}(\Delta) \leq n + p - 1$, *then for any* $L \in \operatorname{Amp}(X)$, *the logarithmic tangent sheaf* $\mathscr{T}_X(-\log D)$ *is unstable with respect to* L.

Proof. If $1 + p \leq \operatorname{card}(\Delta) \leq n + p - 1$, by using

$$\operatorname{card}(\Sigma(1)) = n + p = \operatorname{card}(\Delta) + \operatorname{card}(\Sigma(1) \setminus \Delta),$$

we get $1 \leq \operatorname{card}(\Sigma(1) \setminus \Delta) \leq n - 1$; we can conclude with Proposition 3.1.12. □

Remark 3.1.14. By Corollary 3.1.6, if $\operatorname{card}(\Delta) = n + p$, $\mathscr{T}_X(-\log D)$ is semistable with respect to any polarization.

From now on, we will study the (semi)stability of $\mathscr{T}_X(-\log D)$ only in the case where $1 \leq \operatorname{card}(\Delta) \leq p = \operatorname{rk}\operatorname{Cl}(X)$ and $p \in \{1, 2\}$.

3.2. Stability of equivariant logarithmic tangent sheaves

3.2.1. Stability on weighted projective spaces. In this section, we assume that $X = \mathbb{P}(q_0, \ldots, q_n)$ with $\gcd(q_0, \ldots, q_n) = 1$. We use the notation of Section 2.2.1 and we denote by D_i the divisor of X corresponding to the ray $\operatorname{Cone}(u_i)$. Let $A_i = \{0, \ldots, n\} \setminus \{i\}$ for $i \in \{0, \ldots, n\}$. We set $\mathscr{E} = \mathscr{T}_X(-\log D_i)$.

Proposition 3.2.1. *Let* $L \in \operatorname{Amp}(X)$. *The sheaf* $\mathscr{E}$ *is polystable with respect to* L *if and only if there is* $q \in \mathbb{N}^*$ *such that for all* $j \in A_i$, $q_j = q$.

Proof. We first show that $q_i D_j \sim_{\text{lin}} q_j D_i$. For $m = (a_0, \ldots, a_n) \in M$ defined by $a_i = q_j$, $a_j = -q_i$ and $a_k = 0$ if $k \in A_i \setminus \{j\}$, we get $\operatorname{div}(\chi^m) = q_j D_i - q_i D_j$. Hence, $q_i D_j \sim_{\text{lin}} q_j D_i$. Therefore, for any $L \in \operatorname{Amp}(X)$, $q_i \deg_L(D_j) = q_j \deg_L(D_i)$.

The assumptions of Proposition 3.1.9 are verified. Hence, $\mathscr{E} = \bigoplus_{j \in A_i} \mathscr{O}_X(D_j)$. By Equation (2.21), we get

$$\mu_L(\mathscr{O}_X(D_j)) = \deg_L(D_j) = \frac{q_j}{q_i} \deg_L(D_i).$$

If $\mathscr{E}$ is polystable with respect to L, there is $r \in \mathbb{Q}$ such that for all $j \in A_i$, $q_j = r\, q_i$. Hence, we have the existence of $q \in \mathbb{N}^*$ such that for all $j \in A_i$, $q_j = q$. For the converse, if for all $j \in A_i$, we have $q_j = q$, then $\mathscr{E}$ is polystable. □

According to Proposition 2.3.15, we get :

Corollary 3.2.2. *For all* $i \in \{0, \ldots, n\}$, $\operatorname{sStab}(\mathscr{T}_X(-\log D_i)) \neq \varnothing$ *if and only if there exists* $q \in \mathbb{N}^*$ *such that for all* $j \in A_i$, $q_j = q$. *Moreover, if for all* $j \in A_i$, $q_j = q$, *then*

$$\varnothing = \operatorname{Stab}(\mathscr{T}_X(-\log D_i)) \subsetneq \operatorname{sStab}(\mathscr{T}_X(-\log D_i)) = \operatorname{Amp}(X).$$

3.2.2. Condition of stability on toric varieties of Picard rank two. In this part, we adapt some results of [14, Section 4] for the study of the stability of $\mathscr{T}_X(-\log D)$ when $X = \mathbb{P}\left(\mathscr{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^r \mathscr{O}_{\mathbb{P}^s}(a_i)\right)$ with $0 \leq a_1 \leq \ldots \leq a_r$. We use notation of Sections 2.2.2 and 2.2.3. The following lemma will be useful in the proof of Proposition 3.2.5 which is the main result of this part. Let $z \in \{0, \ldots, r-1\}$ such that $a_z = 0$ and $a_{z+1} > 0$, we have:

Lemma 3.2.3 ([14, Lemma 4.2]). *Let* $I' \subseteq \{0, 1 \ldots, r\}$ *and* $G = \operatorname{Span}(v_i : i \in I')$. *The vector* $a_1 v_1 + \ldots + a_r v_r$ *belongs to* G *if and only if*

i. $\{z+1, \ldots, r\} \subseteq I'$ *or*
ii. $\{0, \ldots, z\} \subseteq I'$, $\operatorname{card}(\{z+1, \ldots, r\} \setminus I') \geq 1$ *and* $a_i = a_j$ *for all* $i, j \in \{z+1, \ldots, r\} \setminus I'$.

Let P be the polytope corresponding to the $\mathbb{Q}$-polarized toric variety (X, L) where $L = \pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ with $\nu \in \mathbb{Q}_{>0}$.

Notation 3.2.4. For all $i \in \{0, 1, \dots, r\}$, we set $\mathrm{V}_i = \mathrm{vol}(P^{v_i})$. As for all $j \in \{1, \dots, s\}$, $\mathrm{vol}(P^{w_j}) = \mathrm{vol}(P^{w_0})$, we set $\mathrm{W} = \mathrm{vol}(P^{w_0})$.

Let $\Delta \subseteq \Sigma(1)$ and D be an invariant reduced Weil divisor of X given by $D = \sum_{\rho \in \Delta} D_\rho$. We set

$$\begin{aligned} I_\Sigma &= \{\mathrm{Cone}(v_0), \dots, \mathrm{Cone}(v_r)\} , \\ J_\Sigma &= \{\mathrm{Cone}(w_0), \dots, \mathrm{Cone}(w_s)\} , \\ I &= \{i \in \{0, 1, \dots, r\} : \mathrm{Cone}(v_i) \in I_\Sigma \setminus (I_\Sigma \cap \Delta)\} \quad \text{and} \\ J &= \{j \in \{0, 1, \dots, s\} : \mathrm{Cone}(w_j) \in J_\Sigma \setminus (J_\Sigma \cap \Delta)\} . \end{aligned}$$

According to Lemma 3.1.11, to study the stability of $\mathscr{E} = T_X(-\log D)$ with respect to L, it suffices to compare $\mu_L(\mathscr{E})$ and $\mu_L(\mathscr{E}_G)$ when $G = \mathrm{Span}(v_i, w_j : i \in I', j \in J')$ with $I' \subseteq I$, $J' \subseteq J$ and $1 \leq \dim G < (r + s)$. By Proposition 2.1.36, (3.4) and (3.5), we get

$$\mu_L(\mathscr{E}) = \frac{1}{r+s} \left(\sum_{i \in I} \mathrm{V}_i + \mathrm{card}(J) \cdot \mathrm{W} \right)$$

and

$$\mu_L(\mathscr{E}_G) = \frac{1}{\dim G} \left(\sum_{i \in I'} \mathrm{V}_i + \mathrm{card}(J') \cdot \mathrm{W} \right) .$$

Here is a version of [14, Proposition 4.1] for logarithmic tangent bundles.

Proposition 3.2.5. *The logarithmic tangent bundle $\mathscr{E} = \mathscr{T}_X(-\log D)$ is stable (resp. semistable) with respect to $L = \pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $\mu_L(\mathscr{E})$ is greater than (resp. greater than or equal to) the maximum of*

1. V_{i_0} *where* $i_0 = \min I$ *if* $I \neq \varnothing$;
2. $\frac{1}{r'} \left(\sum_{i \in I} \mathrm{V}_i\right)$, *if* $r' = \dim \mathrm{Span}(v_i : i \in I) \neq 0$;
3. $\frac{\mathrm{card}(J) \cdot \mathrm{W}}{s'}$, *if* $0 < s' = \dim \mathrm{Span}(w_j : j \in J) < r + s$;
4. $\frac{1}{s+k} \left(\sum_{i \in I'} \mathrm{V}_i + (s+1)\mathrm{W}\right)$, *if* $\mathrm{card}(J') = s+1, k = \mathrm{card}(I') < r$ *and* $\{z+1, \dots, r\} \subseteq I' \subseteq I$;
5. $\frac{1}{s+k} \left(\sum_{i \in I'} \mathrm{V}_i + (s+1)\mathrm{W}\right)$, *if* $\mathrm{card}(J') = s+1$, $k = \mathrm{card}(I') < r$ *and* $I' \subseteq I$ *such that the condition ii. of Lemma 3.2.3 is verified.*

Proof. We set $G = \mathrm{Span}(v_i, w_j : i \in I', j \in J')$ where $I' \subseteq I$ and $J' \subseteq J$. In Proposition 3.2.5, each point corresponds to a value of $\mu_L(\mathscr{E}_G)$ for some G. In particular, (1) corresponds to $G = \mathrm{Span}(v_{i_0})$, (2) corresponds to $G = \mathrm{Span}(v_i : i \in I)$ and (3) corresponds to $G = \mathrm{Span}(w_j : j \in J)$.

If $\mathrm{card}(J') = 0$, then for $\varnothing \subsetneq I' \subseteq I$, we have $\dim G \leq r$ and

$$\mu_L(\mathscr{E}_G) = \frac{1}{\dim G} \sum_{i \in I'} \mathrm{V}_i ;$$

this number is less than or equal to the maximum of the numbers given in (1) and (2).

If $\mathrm{card}(I') = 0$, then for $\varnothing \subsetneq J' \subseteq J$ such that $\dim G < r + s$, we have

$$\mu_L(\mathscr{E}_G) = \frac{\mathrm{card}(J') \cdot \mathrm{W}}{\dim G} \ ;$$

this number is less than or equal to that given in (3).

If $\mathrm{card}(I') = r + 1$, then $\dim G < r + s$ if and only if $s' = \mathrm{card}(J') < s$. If $1 \leq s' < s$, then

$$\mu_L(\mathscr{E}_G) = \frac{1}{r + s'} \left(\sum_{i \in I'} \mathrm{V}_i + s'\mathrm{W} \right) \leq \max \left(\frac{1}{r} \sum_{i \in I'} \mathrm{V}_i, \mathrm{W} \right) .$$

If $1 \leq \mathrm{card}(I') \leq r$, $1 \leq \mathrm{card}(J') \leq s$ and $\dim G < r + s$, then $\mu_L(\mathscr{E}_G)$ is less than or equal to the maximum of numbers given in (1), (2) and (3).

It remains to study the case where $\mathrm{card}(J') = s + 1$ and $1 \leq \mathrm{card}(I') < r$ (because if $\mathrm{card}(I') \geq r$, we have $\dim G = r + s$). We will treat it in two cases.

First case : $a_r = 0$. For all $i \in \{1, \ldots, r\}$, $\mathrm{V}_i = \mathrm{V}_0$. If $r' = \mathrm{card}(I')$ and $1 \leq r' < r$, then

$$\mu_L(\mathscr{E}_G) = \frac{1}{r' + s} \left(\sum_{i \in I'} \mathrm{V}_i + (s+1)\mathrm{W} \right) \leq \max \left(\mathrm{V}_0, \frac{(s+1)\mathrm{W}}{s} \right) .$$

Second case : $a_r > 0$. We set $r' = \mathrm{card}(I')$. If I' satisfies the first (resp. second) condition of Lemma 3.2.3, then the value of $\mu_L(\mathscr{E}_G)$ is given in the point (4) (resp. (5)). If I' does not satisfy the conditions of Lemma 3.2.3, then $\dim G = r' + (s + 1)$. Moreover, if $r' + (s + 1) < r + s$, then the number $\mu_L(\mathscr{E}_G)$ is less than or equal to the maximum of the numbers given in (1) and (3). ☐

Remark 3.2.6. If $a_1 = \ldots = a_r = 0$, to check the stability of $\mathscr{E}$ with respect to L, it is enough to compare $\mu_L(\mathscr{E})$ with the numbers given by the points 1, 2 and 3 of Proposition 3.2.5. In that case, we have

$$\mathrm{W} = \binom{s+r-1}{s-1} \nu^{s-1} \quad \text{and} \quad \mathrm{V}_i = \binom{s+r-1}{s} \nu^{s}. \tag{3.7}$$

If $(a_1, \ldots, a_r) \neq (0, \ldots, 0)$, the results below will help us to determine if $\mathscr{E}$ is unstable with respect to L without having to check each point of Proposition 3.2.5. Let $z \in \{0, 1, \ldots, r-1\}$ such that $a_z = 0$ and $a_{z+1} > 0$ where $a_0 = 0$. Let $k \in \{0, \ldots, s\}$. We set

$$\mathrm{V}_{0k} = \sum_{d_{z+1} + \ldots + d_r = s-k} a_{z+1}^{d_{z+1}} \cdots a_r^{d_r} \quad \text{and} \quad \mathrm{W}_k = \sum_{d_{z+1} + \ldots + d_r = s-1-k} a_{z+1}^{d_{z+1}} \cdots a_r^{d_r}$$

where $\mathrm{W}_s = 0$. For $i \in \{z + 1, \ldots, r\}$, we set

$$\mathrm{V}_{ik} = \sum_{\substack{d_{z+1} + \ldots + d_{i-1} \\ + d_{i+1} + \ldots + d_r = s-k}} a_{z+1}^{d_{z+1}} \cdots a_{i-1}^{d_{i-1}} a_{i+1}^{d_{i+1}} \cdots a_r^{d_r}$$

and for $i \in \{1, \ldots, z\}$, we set $\mathrm{V}_{ik} = \mathrm{V}_{0k}$.

Remark 3.2.7. If $r = 1$, we set $\mathrm{V}_{1s} = 1$ and for $k \in \{0, \ldots, s-1\}$, $\mathrm{V}_{1k} = 0$. We have $\mathrm{W}_{s-1} = 1$ and $\mathrm{V}_{is} = 1$ for any $i \in \{0, \ldots, r\}$.

Lemma 3.2.8. *For all* $i \in \{1, \ldots, r\}$, $\mathrm{V}_0 = a_i \mathrm{W} + \mathrm{V}_i$.

Proof. To show the lemma, it suffices to show that: for any $k \in \{0, \dots, s-1\}$, $a_i \mathrm{W}_k + \mathrm{V}_{ik} = \mathrm{V}_{0k}$. If $i \in \{1, \dots, z\}$, the equality is true because $a_i = 0$. We assume that $i \in \{z+1, \dots, r\}$, we have

$$\begin{aligned} \mathrm{V}_{0k} &= \sum_{d_{z+1}+\ldots+d_r = s-k} a_{z+1}^{d_{z+1}} \cdots a_r^{d_r} \\ &= \sum_{\substack{d_{z+1}+\ldots+d_r = s-k \\ d_i = 0}} a_{z+1}^{d_{z+1}} \cdots a_r^{d_r} + \sum_{\substack{d_{z+1}+\ldots+d_r = s-k \\ d_i \geq 1}} a_{z+1}^{d_{z+1}} \cdots a_r^{d_r} \end{aligned}$$

The first term of the second line corresponds to the number V_{ik} and the second to $a_i \mathrm{W}_k$ (it suffices to replace d_i by $d_i' + 1$). Hence, $\mathrm{V}_{0k} = \mathrm{V}_{ik} + a_i \mathrm{W}_k$. □

Lemma 3.2.9. *Let* $(a_1, \dots, a_r) \neq (0, \dots, 0)$

1. *If* $a_r \geq 2$*, then* $s\mathrm{V}_0 - (s+1)\mathrm{W} \geq s\mathrm{V}_r$*.*
2. *If* $r \geq 2$ *and* $i \in \{1, \dots, r-1\}$ *with* $a_i < a_r$*, then* $\mathrm{V}_i - \mathrm{W} \geq \mathrm{V}_r$*.*

Proof. If $a_r \geq 2$, then $\left(s - \dfrac{s+1}{a_r}\right) = \dfrac{a_r s - (s+1)}{a_r} \geq \dfrac{2s - (s+1)}{a_r} \geq 0$ because $s \geq 1$. Hence,

$$\begin{aligned} s\mathrm{V}_0 - (s+1)\mathrm{W} &= s\mathrm{V}_0 - \frac{s+1}{a_r}(\mathrm{V}_0 - \mathrm{V}_r) \\ &= \left(s - \frac{s+1}{a_r}\right)\mathrm{V}_0 + \frac{s+1}{a_r}\mathrm{V}_r \\ &\geq \left(s - \frac{s+1}{a_r}\right)\mathrm{V}_r + \frac{s+1}{a_r}\mathrm{V}_r = s\mathrm{V}_r\,. \end{aligned}$$

As $\mathrm{V}_0 = a_i \mathrm{W} + \mathrm{V}_i = a_r \mathrm{W} + \mathrm{V}_r$, we get $\mathrm{V}_i = (a_r - a_i)\mathrm{W} + \mathrm{V}_r$. If $a_r > a_i$, then $a_r - a_i \geq 1$; therefore $\mathrm{V}_i \geq \mathrm{W} + \mathrm{V}_r$. □

3.3. Stability on smooth toric varieties of Picard rank two

3.3.1. Stability of logarithmic tangent bundles on a product of projective spaces. We assume that $a_1 = \ldots = a_r = 0$. Let Σ be the fan of X, we have $X \cong \mathbb{P}^s \times \mathbb{P}^r$. We denote by $\pi_1 : X \longrightarrow \mathbb{P}^s$ and $\pi_2 : X \longrightarrow \mathbb{P}^r$ the projection maps. Let

$$\{D'_{w_j} : 0 \leq j \leq s\} \quad \text{and} \quad \{D'_{v_i} : 0 \leq i \leq r\}$$

be respectively the set of invariant divisors of $\mathbb{P}^s$ and $\mathbb{P}^r$ such that for any $j \in \{0, \dots, s\}$ and any $i \in \{0, \dots, r\}$,

$$\pi_1^* D'_{w_j} = D_{w_j} \quad \text{and} \quad \pi_2^* D'_{v_i} = D_{v_i}$$

where D_{v_i} and D_{w_j} are the invariant divisors of X defined in Section 2.2.2. We will show that:

Theorem 3.3.1. *Let* $i \in \{0, \dots, r\}$ *and* $j \in \{0, \dots, s\}$*. Then:*

1. $\mathscr{T}_X(-\log D_{v_i})$ *is polystable with respect to* $\pi^* \mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ *if and only if* $\nu = \dfrac{s+1}{r}$;
2. $\mathscr{T}_X(-\log D_{w_j})$ *is polystable with respect to* $\pi^* \mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ *if and only if* $\nu = \dfrac{s}{r+1}$;
3. $\mathscr{T}_X(-\log(D_{v_i} + D_{w_j}))$ *is polystable with respect to* $\pi^* \mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ *if and only if* $\nu = \dfrac{s}{r}$.

Let $\Delta \subseteq \Sigma(1)$ and $D = \sum_{\rho\in\Delta} D_\rho$ be an invariant reduced divisor of X. As $\mathscr{T}_X \cong \pi_1^*\mathscr{T}_{\mathbb{P}^s} \oplus \pi_2^*\mathscr{T}_{\mathbb{P}^r}$, for any $\Delta \subseteq \Sigma(1)$ such that $|\Delta| \in \{1,2\}$, the logarithmic tangent sheaf $\mathscr{T}_X(-\log D)$ is decomposable. The proof of Theorem 3.3.1 will then rely on this lemma.

Lemma 3.3.2. *Let $i, i' \in \{0,\dots,r\}$ and $j, j' \in \{0,\dots,s\}$ such that $i \neq i'$ and $j \neq j'$. Then:*

1. $\mathscr{T}_X(-\log(D_{v_i} + D_{v_{i'}})) \cong \pi_1^*\mathscr{T}_{\mathbb{P}^s} \oplus \pi_2^*\mathscr{T}_{\mathbb{P}^r}(-\log(D'_{v_i} + D'_{v_{i'}}))$.
2. $\mathscr{T}_X(-\log(D_{w_j} + D_{w_{j'}})) \cong \pi_1^*\mathscr{T}_{\mathbb{P}^s}(-\log(D'_{w_j} + D'_{w_{j'}})) \oplus \pi_2^*\mathscr{T}_{\mathbb{P}^r}$.
3. $\mathscr{E} = \mathscr{T}_X(-\log D_{v_i})$ *satisfies*

$$\mathscr{E} \cong \pi_1^*\mathscr{T}_{\mathbb{P}^s} \oplus \left(\bigoplus_{k=0,\, k\neq i}^{r} \pi_2^*\mathscr{O}_{\mathbb{P}^r}(D'_{v_k}) \right).$$

4. $\mathscr{E} = \mathscr{T}_X(-\log D_{w_j})$ *satisfies*

$$\mathscr{E} \cong \left(\bigoplus_{k=0,\, k\neq j}^{s} \pi_1^*\mathscr{O}_{\mathbb{P}^s}(D'_{w_k}) \right) \oplus \pi_2^*\mathscr{T}_{\mathbb{P}^r}.$$

Proof. We will only show the point 3. Let Σ_1 be fan of $\mathbb{P}^s$. We denote by ρ_ℓ the ray of Σ_1 corresponding to the divisor D'_{w_ℓ}. According to Example 2.3.11, the family of filtrations $(F, \{F^{\rho_\ell}(j)\})$ of $\mathscr{T}_{\mathbb{P}^s}$ is given by

$$F^{\rho_\ell}(j) = \begin{cases} 0 & \text{if } j < -1 \\ \mathrm{Span}(w_\ell) & \text{if } j = -1 \\ \mathrm{Span}(w_1,\dots,w_s) & \text{if } j > -1 \end{cases}.$$

By Proposition 4.1.1 (the map ϕ is the projection $\mathbb{Z}^s \times \mathbb{Z}^r \longrightarrow \mathbb{Z}^s$), the family of filtrations of $\pi_1^*\mathscr{T}_{\mathbb{P}^s}$ is given by

$$\widetilde{F}^{\rho_{w_\ell}}(j) = \begin{cases} 0 & \text{if } j < -1 \\ \mathrm{Span}(w_\ell) & \text{if } j = -1 \\ \mathrm{Span}(w_1,\dots,w_s) & \text{if } j > -1 \end{cases}$$

and

$$\widetilde{F}^{\rho_{v_k}}(j) = \begin{cases} 0 & \text{if } j < 0 \\ \mathrm{Span}(w_1,\dots,w_s) & \text{if } j \geq 0 \end{cases}$$

where $\rho_{v_k} = \mathrm{Cone}(v_k)$ and $\rho_{w_\ell} = \mathrm{Cone}(w_\ell)$ are the rays of Σ. As $\pi_2^*\mathscr{O}_{\mathbb{P}^r}(D'_{v_k}) \simeq \mathscr{O}_X(D_{v_k})$, by Example 2.3.10 the family of filtrations $(G, \{G^\rho(j)\})$ of $\pi_2^*\mathscr{O}_{\mathbb{P}^r}(D'_{v_k})$ is given by

$$G^\rho(j) = \begin{cases} 0 & \text{if } j < 0 \\ \mathrm{Span}(v_k) & \text{if } j \geq 0 \end{cases}$$

if $\rho \neq \rho_{v_k}$ and

$$G^{\rho_{v_k}}(j) = \begin{cases} 0 & \text{if } j < -1 \\ \mathrm{Span}(v_k) & \text{if } j \geq -1 \end{cases}.$$

Using the Equation (3.3), the family of filtrations of the sheaf on the right side of the decomposition is equal to the family of filtrations of $\mathscr{T}_X(-\log D_{v_i})$. ☐

In Table 3.1, we give the values of ν for which $\mathscr{E} = \mathscr{T}_X(-\log D)$ is semistable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$. We use the fact that a direct sum of vector bundles is semistable if and only if each summand is semistable with the same slope (cf. Proposition 2.3.15). We recall that by Equation (3.7), $\mathrm{V}_i = \dfrac{r\nu}{s}\mathrm{W}$ for any $i \in \{0,\dots,r\}$. We set $\mathrm{V} = \dfrac{r\nu}{s}\mathrm{W}$.

Divisor D	$\mathrm{sStab}(\mathscr{E})$	References
D_{v_i} , $0 \leq i \leq r$	$\nu = \dfrac{s+1}{r}$	Theorem 3.3.1
D_{w_j} , $0 \leq j \leq s$	$\nu = \dfrac{s}{r+1}$	Theorem 3.3.1
$D_{v_j} + D_{w_j}$	$\nu = \dfrac{s}{r}$	Theorem 3.3.1
$D_{v_i} + D_{v_j}$, $0 \leq i < j \leq r$	$\varnothing$	Proposition 3.3.3
$D_{w_i} + D_{w_j}$, $0 \leq i < j \leq s$	$\varnothing$	Proposition 3.3.3

Table 3.1: Stability of $\mathscr{T}_X(-\log D)$ when $a_1 = \ldots = a_r = 0$

Proof of Theorem 3.3.1. We start with $\mathscr{E} = \mathscr{T}_X(-\log D_{v_i})$. We use the point 3 of Lemma 3.3.2. By (2.21), we have

$$\mu_L(\pi_2^*\mathscr{O}_{\mathbb{P}^r}(D'_{v_k})) = \deg_L(D_{v_k}) = \mathrm{V}$$

for any $k \in \{0, \ldots, r\} \setminus \{i\}$. The first Chern class of $\mathscr{T}_{\mathbb{P}^s}$ is given by

$$c_1(\mathscr{T}_{\mathbb{P}^s}) = \sum_{k=0}^{s} D'_{w_k}.$$

Therefore

$$\pi_1^* c_1(\mathscr{T}_{\mathbb{P}^s}) = \sum_{k=0}^{s} D_{w_k} \quad \text{and} \quad \mu_L(\pi_1^*\mathscr{T}_{\mathbb{P}^s}) = \frac{1}{s}\sum_{k=0}^{s} \deg_L(D_{w_k}) = \frac{s+1}{s}\mathrm{W}.$$

As $\pi_1^*\mathscr{T}_{\mathbb{P}^s}$ is stable with respect to L, we deduce that $\mathscr{E}$ is polystable with respect to L if and only if $\frac{s+1}{s}\mathrm{W} = \mathrm{V} = \frac{r\nu}{s}\mathrm{W}$, i.e. $\nu = \frac{s+1}{r}$.

If $\mathscr{E} = \mathscr{T}_X(-\log D_{w_j})$, we use the point 4 of Lemma 3.3.2. We have

$$\mu_L(\pi_1^*\mathscr{O}_{\mathbb{P}^s}(D'_{w_k})) = \mathrm{W} \quad \text{and} \quad \mu_L(\pi_2^*\mathscr{T}_{\mathbb{P}^r}) = \frac{r+1}{r}\mathrm{V}.$$

Hence, $\mathscr{E}$ is polystable with respect to L if and only if $\frac{r+1}{r}\mathrm{V} = \mathrm{W} = \frac{s}{r\nu}\mathrm{V}$, i.e. $\nu = \frac{r}{s+1}$.

We now consider the case $\mathscr{E} = \mathscr{T}_X(-\log(D_{v_i} + D_{w_j}))$. By Proposition 3.1.9, we have

$$\mathscr{E} = \left(\bigoplus_{k=0,\, k\neq j}^{s} \mathscr{O}_X(D_{w_k})\right) \oplus \left(\bigoplus_{l=0,\, l\neq i}^{r} \mathscr{O}_X(D_{v_l})\right).$$

As $\deg_L(\mathscr{O}_X(D_{w_k})) = \mathrm{W}$ for any $k \in \{0, \ldots, s\} \setminus \{j\}$ and $\deg_L(\mathscr{O}_X(D_{v_l})) = \mathrm{V}$ for any $l \in \{0, \ldots, r\} \setminus \{i\}$, we deduce that $\mathscr{E}$ is polystable with respect to L if and only if $\mathrm{W} = \mathrm{V} = \frac{r\nu}{s}\mathrm{W}$, i.e. $\nu = \frac{s}{r}$. □

Proposition 3.3.3. *Let $i, i' \in \{0, \ldots, r\}$ and $j, j' \in \{0, \ldots, s\}$ such that $i \neq i'$ and $j \neq j'$. For any $L \in \mathrm{Amp}(X)$, the logarithmic tangent bundles $\mathscr{T}_X(-\log(D_{v_i} + D_{v_{i'}}))$ and $\mathscr{T}_X(-\log(D_{w_j} + D_{w_{j'}}))$ are not semistable with respect to L.*

Proof. Let $\mathscr{E} = \mathscr{T}_X(-\log(D_{v_i} + D_{v_{i'}}))$ and $L \in \mathrm{Amp}(X)$. We use the point 1 of Lemma 3.3.2. If $r \geq 2$, then $\mathscr{T}_{\mathbb{P}^r}(-\log(D'_{v_i} + D'_{v_{i'}}))$ is unstable with respect to $L_{|\mathbb{P}^r}$ by Corollary 3.1.13. Therefore, $\mathscr{E}$ is unstable with respect to L. If $r = 1$, we have $\mathscr{T}_{\mathbb{P}^r}(-\log(D'_{v_i} + D'_{v_{i'}})) \cong \mathscr{O}_{\mathbb{P}^r}$. As

$$\mu_L(\pi_2^*\mathscr{O}_{\mathbb{P}^r}) = 0 \quad \text{and} \quad \mu_L(\pi_1^*\mathscr{T}_{\mathbb{P}^s}) = \frac{s+1}{s}\mathrm{W} \neq 0,$$

we deduce that $\mathscr{E}$ is unstable with respect to L. □

Remark 3.3.4. According to (2.11) and (2.12), when $a_1 = \ldots = a_r = 0$, we have:

$$D_{v_i} \sim_{\text{lin}} D_{v_0} \ , \ D_{w_j} \sim_{\text{lin}} D_{w_0} \quad \text{and} \ -K_X \sim_{\text{lin}} (s+1)D_{w_0} + (r+1)D_{v_0}.$$

In each point of Theorem 3.3.1, we see that $\mathscr{T}_X(-\log D)$ is polystable with respect to L if and only if $L \cong \mathscr{O}_X(-\alpha\,(K_X + D))$ with $\alpha \in \mathbb{N}^*$.

Case where varieties are not products of projective spaces

We now study the stability of $\mathscr{T}_X(-\log D)$ when $X = \mathbb{P}\left(\mathscr{O}_{\mathbb{P}^s} \oplus \bigoplus_{i=1}^r \mathscr{O}_{\mathbb{P}^s}(a_i)\right)$ with $a_r \geq 1$. Let $\Delta \subseteq \Sigma(1)$ and $D = \sum_{\rho\in\Delta} D_\rho$. By Corollary 3.1.13, we will only study the case where $\operatorname{card}(\Delta) \in \{1, 2\}$. The case $\operatorname{card}(\Delta) = 0$ was treated by Hering-Nill-Süss in [14] and Dasgupta-Dey-Khan in [4]. In the following theorem, we give a classification of pairs (X, D) such that $\operatorname{Stab}(\mathscr{T}_X(-\log D)) \neq \varnothing$ or $\operatorname{sStab}(\mathscr{T}_X(-\log D)) \neq \varnothing$. More precisely, we give the values of ν for which $\mathscr{E} = \mathscr{T}_X(-\log D)$ is (semi)stable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ in the Tables 3.2, 3.3, 3.4 and the references therein.

Theorem 3.3.5. *Let $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(a_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^s}(a_r))$ with $(a_1, \ldots, a_r) \neq (0, \ldots, 0)$ and D a reduced invariant divisor of X. Then:*

1. *There is $L \in \operatorname{Amp}(X)$ such that $\mathscr{T}_X(-\log D)$ is stable with respect to L if and only if:*
   i. *$(a_1, \ldots, a_r) = (0, \ldots, 0, 1)$ and $D = D_{v_r}$, or*
   ii. *$a_1 = \ldots = a_r$ with $(r-1)a_r < (s+1)$ and $D = D_{v_0}$.*
2. *There is $L \in \operatorname{Amp}(X)$ such that $\mathscr{T}_X(-\log D)$ is polystable with respect to L if and only if:*
   i. *$a_1 = \ldots = a_r$ with $(r-1)a_r < s$ and*
   $$D \in \{D_{v_0} + D_{w_j} : 0 \leq j \leq s\} \cup \{D_{v_0} + D_{v_i} : 1 \leq i \leq r\},$$
   ii. *or $1 \leq a_1 < a_2 = \ldots = a_r$ and $D = D_{v_0} + D_{v_1}$ with $\ell(s) > 0$ where $\ell : \mathbb{N}^* \longrightarrow \mathbb{R}$ is the map given by*
   $$\ell(p) = \sum_{j=0}^{p-1} \binom{j+r-2}{j} \left(1 - \frac{a_r(r-2)}{j+1}\right) \left(\frac{a_r}{a_1}\right)^j - a_1. \tag{3.8}$$
3. *Otherwise, the sheaf $\mathscr{T}_X(-\log D)$ is unstable with respect to any polarization.*

Remark 3.3.6. We will show in Section 3.3.5 that if $p_0 \in \mathbb{N}$ satisfies $\ell(p_0) > 0$, then for any $p \geq p_0$, $\ell(p) > 0$. If $r = 2$, the condition $\ell(s) > 0$ is equivalent to $s > \dfrac{\ln(1 + a_r - a_1)}{\ln(a_r) - \ln(a_1)}$.

Before proving this theorem, we give a similar version of Lemma 3.3.2. We recall that $a_0 = 0$.

Lemma 3.3.7. *We assume that $a_r \geq 1$.*

1. *If $i \in \{0, \ldots, r\}$ and $j \in \{0, \ldots, s\}$, then $\mathscr{E} = \mathscr{T}_X(-\log(D_{v_i} + D_{w_j}))$ is decomposable and*
   $$\mathscr{E} \cong \left(\bigoplus_{l=0,\, l\neq j}^{s} \mathscr{O}_X(D_{w_l})\right) \oplus \left(\bigoplus_{k=0,\, k\neq i}^{r} \mathscr{O}_X(D_{v_k})\right).$$
2. *If $i, j \in \{0, \ldots, s\}$ with $i \neq j$, then the sheaves $\mathscr{F} = \mathscr{T}_X(-\log(D_{w_i} + D_{w_j}))$ and $\mathscr{E} = \mathscr{T}_X(-\log D_{w_j})$ are decomposable and*
   $$\mathscr{F} \cong \left(\bigoplus_{k=0,\, k\notin\{i,j\}}^{s} \mathscr{O}_X(D_{w_k})\right) \oplus \mathscr{O}_X \oplus \mathscr{F}_G \quad , \quad \mathscr{E} \cong \left(\bigoplus_{k=0,\, k\neq j}^{s} \mathscr{O}_X(D_{w_k})\right) \oplus \mathscr{E}_G$$
   where $G = \operatorname{Span}(v_0, \ldots, v_r)$.

3. *If $D = D_{v_i} + D_{v_j}$ for $0 \leq i < j \leq r$, then the sheaf $\mathscr{E} = \mathscr{T}_X(-\log D)$ is decomposable. If $a_i < a_j$, then*

$$\mathscr{E} \cong \left(\bigoplus_{l=0}^{s} \mathscr{O}_X(D_{w_l}) \right) \oplus \left(\bigoplus_{k=0,\, k \notin \{i,j\}}^{r} \mathscr{O}_X(D_{v_k}) \right).$$

If $a_i = a_j$, then

$$\mathscr{E} \cong \mathscr{E}_G \oplus \mathscr{O}_X$$

where $G = \mathrm{Span}(w_l, v_k : l \in \{0,\dots,s\}, k \in \{0,\dots,r\} \setminus \{i,j\})$.

Remark 3.3.8. If $D \in \{D_{v_i} : 0 \leq i \leq r\}$, then the sheaf $\mathscr{E} = \mathscr{T}_X(-\log D)$ is not always decomposable. In particular, if we assume that $r = 2$, $s = 1$, $a_1 = 0$ and $a_2 = 1$, then $\mathscr{E} = \mathscr{T}_X(-\log D_{v_1})$ is decomposable with $\mathscr{E} = \mathscr{E}_F \oplus \mathscr{E}_G$ where $F = \mathrm{Span}(v_2, w_1)$ and $G = \mathrm{Span}(v_0)$. But $\mathscr{F} = \mathscr{T}_X(-\log D_{v_2})$ is not decomposable.

Let $L = \pi^* \mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ be a $\mathbb{Q}$-polarization of X. We recall that, for any $j \in \{0,\dots,s\}$ and any $i \in \{0,\dots,r\}$,

$$\deg_L(D_{w_k}) = \mathrm{W} \quad \text{and} \quad \deg_L(D_{v_i}) = \mathrm{V}_i$$

where $\mathrm{W}, \mathrm{V}_0, \dots, \mathrm{V}_r$ are polynomials of ν defined on Section 2.2.3. To check the stability of the logarithmic tangent sheaves, we will use the description of its saturated subsheaves given in Notation 3.1.7 and also the description of invertible sheaves given in Example 2.3.10. We will also need the sign changes rule of Descartes [29, Chapter 5, Section 4.3].

Theorem 3.3.9 (Descartes). *Let $\mathrm{P} = c_n X^n + c_{n-1} X^{n-1} + \ldots + c_0$ be a polynomial with real coefficients where $c_n\, c_0 \neq 0$. Let p the number of sign changes in the sequence $(c_0, \dots, c_n)$ of its coefficients and q the numbers of positive real roots, counted with their order of multiplicity. Then, there is $m \in \mathbb{N}$ such that $q = p - 2m$.*

3.3.2. Case of divisors coming from the base.

Proposition 3.3.10. *Let $(a_1, \dots, a_r) \neq (0, \dots, 0)$. For any $i, j \in \{0, \dots, s\}$ with $i \neq j$, the sheaves $\mathscr{E} = \mathscr{T}_X(-\log D_{w_j})$ and $\mathscr{F} = \mathscr{T}_X(-\log(D_{w_i} + D_{w_j}))$ are unstable with respect to any polarization.*

Proof. We use the point 2 of Lemma 3.3.7, Proposition 2.3.15 and Equation (2.21). Let $L = \pi^* \mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$. We first consider the sheaf $\mathscr{F}$. As $\mu_L(\mathscr{O}_X) = 0$ and $\mu_L(\mathscr{O}_X(D_{w_k})) = \mathrm{W} \neq 0$, we deduce that $\mathscr{F}$ is not semistable with respect to L.

We now consider the sheaf $\mathscr{E}$. By Lemma 3.2.8, we have $\mathrm{V}_0 = a_r \mathrm{W} + \mathrm{V}_r$. As $a_r \geq 1$, for any $k \in \{0, \dots, s\}$,

$$\mu_L(\mathscr{O}_X(D_{w_k})) = \mathrm{W} < \mathrm{V}_0 = \mu_L(\mathscr{O}_X(D_{v_0})).$$

If $r = 1$, then $\mathscr{E}_G \cong \mathscr{O}_X(D_{v_0} + D_{v_1})$. As $\mu_L(\mathscr{E}_G) = \mathrm{V}_0 + \mathrm{V}_1$, we deduce that $\mathscr{E}$ is not polystable with respect to L. Hence, $\mathscr{E}$ is unstable with respect to L.

We now assume that $r \geq 2$. If the sheaf $\mathscr{E}_G$ is unstable with respect to L, then $\mathscr{E}$ is unstable with respect to L. Otherwise, if $\mathscr{E}_G$ is semistable with respect to L, then

$$\mu_L(\mathscr{O}_X(D_{v_0})) = \mathrm{V}_0 \leq \mu_L(\mathscr{E}_G)$$

because $\mathscr{O}_X(D_{v_0})$ is a subsheaf of $\mathscr{E}_G$. Therefore, $\mu_L(\mathscr{O}_X(D_{w_k})) < \mu_L(\mathscr{E}_G)$. Hence, $\mathscr{E}$ is unstable with respect to L. ☐

Divisor D	Condition on r and a_i	Condition on s	$\mathrm{Stab}(\mathscr{E})$	$\mathrm{sStab}(\mathscr{E})$
D_{w_j} , $0 \leq j \leq s$ Proposition 3.3.10	$r \geq 1$ and $a_r \geq 1$	$s \geq 1$	$\varnothing$	$\varnothing$
D_{v_i} , $1 \leq i \leq r-1$ Proposition 3.3.12	$r \geq 2$ and $a_r \geq 1$	$s \geq 1$	$\varnothing$	$\varnothing$
D_{v_r} Theorem 3.3.13	$r \geq 1, a_r = 1$ and $a_{r-1} = 0$	$s \geq 1$	$0 < \nu < \nu_0$	$0 < \nu \leq \nu_0$
	$r \geq 1$ and ($a_r \geq 2$ or $a_{r-1} \neq 0$)	$s \geq 1$	$\varnothing$	$\varnothing$
D_{v_0} Theorem 3.3.15 Lemma 3.3.14 Theorem 3.3.17	$r = 1$	$s \geq 1$	$0 < \nu < \nu_1$	$0 < \nu \leq \nu_1$
	$r \geq 2$ and $a_1 < a_r$	$s \geq 1$	$\varnothing$	$\varnothing$
	$r \geq 2$ and $a_1 = a_r = a$	$a \geq \frac{s+1}{r-1}$	$\varnothing$	$\varnothing$
		$\frac{s}{r} \leq a < \frac{s+1}{r-1}$	$0 < \nu < \nu_1$	$0 < \nu \leq \nu_1$
		$a\,r < s$	$\nu_2 < \nu < \nu_1$	$\nu_2 \leq \nu \leq \nu_1$

Table 3.2: Stability of $\mathscr{T}_X(-\log D)$ when $a_r \geq 1$

Divisor D	Condition on r and a_i	Condition on s	$\mathrm{sStab}(\mathscr{E})$
$D_{w_i} + D_{w_j}$, $0 \leq i < j \leq s$ Proposition 3.3.10	$r \geq 1$ and $a_r \geq 1$	$s \geq 1$	$\varnothing$
$D_{v_i} + D_{v_j}$, $1 \leq i < j \leq r$ Proposition 3.3.11	$r \geq 2$ and $a_r \geq 1$	$s \geq 1$	$\varnothing$
$D_{v_i} + D_{w_j}$, $j \geq 0$ and $1 \leq i \leq r$ Proposition 3.3.11	$r \geq 1$ and $a_r \geq 1$	$s \geq 1$	$\varnothing$
$D_{v_0} + D_{w_j}$, $0 \leq j \leq s$ Theorem 3.3.15 Lemma 3.3.14 Proposition 3.3.18	$r = 1$	$s \geq 1$	$\nu = \nu_2$
	$r \geq 2$ and $a_1 < a_r$	$s \geq 1$	$\varnothing$
	$r \geq 2$ and $a_1 = a_r = a$	$s \leq a(r-1)$	$\varnothing$
		$s > a(r-1)$	$\nu = \nu_2$
$D_{v_0} + D_{v_i}$, $2 \leq i \leq r$ Lemma 3.3.14 Theorem 3.3.18	$r \geq 2$ and $a_1 < a_r$	$s \geq 1$	$\varnothing$
	$r \geq 2$ and $a_1 = a_r = a$	$s \leq a(r-1)$	$\varnothing$
		$s > a(r-1)$	$\nu = \nu_2$

Table 3.3: Stability of $\mathscr{T}_X(-\log D)$ when $a_r \geq 1$

Divisor D	Condition on r and a_i	Condition on s	$\mathrm{sStab}(\mathscr{E})$
$D_{v_0}+D_{v_1}$	$r=1$	$s\geq 1$	$\nu>0$
	$r\geq 2$ and $0=a_1<a_r$	$s\geq 1$	$\varnothing$
	$r\geq 3$ and $a_2<a_r$	$s\geq 1$	$\varnothing$
Theorem 3.3.15	$r\geq 2$ and $a_1=\ldots=a_r=a$	$s\leq a(r-1)$	$\varnothing$
Proposition 3.3.19		$s>a(r-1)$	$\nu=\nu_2$
Proposition 3.3.18	$r\geq 2$ and $0<a_1<a_2=\ldots=a_r$	$\ell(s)\leq 0$	$\varnothing$
Proposition 3.3.20		$\ell(s)>0$	$\nu=\nu_2$

Table 3.4: Stability of $\mathscr{T}_X(-\log(D_{v_0}+D_{v_1}))$ when $a_r\geq 1$ with ℓ given in (3.8)

3.3.3. Sum of divisors coming from the base and the bundle: first part. In this section, we study the stability of $\mathscr{T}_X(-\log D)$ when $D\in\mathscr{D}$ with

$$\mathscr{D}=\{D_{v_i}:1\leq i\leq r\}\cup\{D_{v_i}+D_{w_j}:1\leq i\leq r\text{ and }0\leq j\leq s\}\\ \cup\{D_{v_i}+D_{v_j}:1\leq i<j\leq r\}\,.$$

Proposition 3.3.11. *Let $r\in\mathbb{N}^*$ and $a_r\geq 1$.*

1. *For any $j\in\{0,\ldots,s\}$ and $i\in\{1,\ldots,r\}$, the sheaf $\mathscr{E}=\mathscr{T}_X(-\log(D_{v_i}+D_{w_j}))$ is unstable with respect to any polarization.*
2. *If $r\geq 2$ and $i,j\in\{1,\ldots,r\}$ with $i\neq j$, then the sheaf $\mathscr{F}=\mathscr{T}_X(-\log(D_{v_i}+D_{v_j}))$ is unstable with respect to any polarization.*

Proof. By the point 1 of Lemma 3.3.7, $\mathscr{E}$ is direct sum of line bundles. As $\mu_L(\mathscr{O}_X(D_{w_k}))=\mathrm{W}<\mathrm{V}_0=\mu_L(\mathscr{O}_X(D_{v_0}))$, we deduce that $\mathscr{E}$ is not semistable with respect to L.

For the sheaf $\mathscr{F}$, we use the point 3 of Lemma 3.3.7. If $a_i=a_j$, then $\mathscr{E}$ is not semistable. Otherwise, if $a_i\neq a_j$, we have $\mu_L(\mathscr{O}_X(D_{w_0}))=\mathrm{W}<\mathrm{V}_0=\mu_L(\mathscr{O}_X(D_{v_0}))$; hence $\mathscr{E}$ is not semistable with respect to L. □

Proposition 3.3.12. *Let $r\geq 2$ and $(a_1,\ldots,a_r)\neq(0,\ldots,0)$. For any $i\in\{1,\ldots,r-1\}$, the logarithmic tangent bundle $\mathscr{E}=\mathscr{T}_X(-\log D_{v_i})$ is unstable with respect to any polarization.*

Proof. For $L\in\mathrm{Amp}(X)$, we have

$$\mu_L(\mathscr{E})=\frac{(s+1)\mathrm{W}+(\mathrm{V}_0+\ldots+\mathrm{V}_{i-1}+\mathrm{V}_{i+1}+\ldots+\mathrm{V}_r)}{r+s}.$$

As by Lemma 3.2.8, we have $\mathrm{V}_0-\mathrm{W}\geq\mathrm{V}_r$, we get

$$\begin{aligned}(r+s)(\mathrm{V}_0-\mu_L(\mathscr{E}))=&(s+1)(\mathrm{V}_0-\mathrm{W})-\mathrm{V}_r\\ &+((r-1)\mathrm{V}_0-(\mathrm{V}_0+\ldots+\mathrm{V}_{i-1}+\mathrm{V}_{i+1}+\ldots+\mathrm{V}_{r-1}))\\ \geq&(s+1)(\mathrm{V}_0-\mathrm{W})-\mathrm{V}_r\\ \geq&(s+1)\mathrm{V}_r-\mathrm{V}_r=s\mathrm{V}_r\,.\end{aligned}$$

Therefore, by Proposition 3.2.5, $\mathscr{E}$ is not semistable with respect to L. □

We now study the stability of $\mathscr{T}_X(-\log D_{v_r})$.

Theorem 3.3.13. *Let* $r \geq 1$ *and* $a_r \geq 1$. *We have* $\mathrm{Stab}(\mathscr{T}_X(-\log D_{v_r})) \neq \varnothing$ *if and only if* $\mathrm{sStab}(\mathscr{T}_X(-\log D_{v_r})) \neq \varnothing$ *if and only if* $a_r = 1$ *and* $a_{r-1} = 0$. *If* $a_r = 1$ *and* $a_{r-1} = 0$, *then the logarithmic tangent bundle* $\mathscr{T}_X(-\log D_{v_r})$ *is stable (resp. semistable) with respect to* $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ *if and only if* $0 < \nu < \nu_0$ *(resp.* $0 < \nu \leq \nu_0$*) where* ν_0 *is the positive root of*

$$\mathrm{P}_0(x) = \sum_{k=0}^{s-1} \binom{s+r-1}{k} x^k - s\binom{s+r-1}{s} x^s \,.$$

Proof. Let $\mathscr{E} = \mathscr{T}_X(-\log D_{v_r})$ and $L = \pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$. We have

$$(r+s)\mu_L(\mathscr{E}) = (s+1)\mathrm{W} + \mathrm{V}_0 + \mathrm{V}_1 + \ldots + \mathrm{V}_{r-1}.$$

If $a_r \geq 2$, by using the first point of Lemma 3.2.9 and the fact that $\mathrm{V}_i \leq \mathrm{V}_0$, we get :

$$(r+s)[\mathrm{V}_0 - \mu_L(\mathscr{E})] = (s\mathrm{V}_0 - (s+1)\mathrm{W}) + r\mathrm{V}_0 - (\mathrm{V}_0 + \ldots + \mathrm{V}_{r-1}) \geq s\mathrm{V}_r \,.$$

By Proposition 3.2.5, $\mathscr{T}_X(-\log D_{v_r})$ is not semistable with respect to L.

If $r \geq 2$ and $a_{r-1} = a_r = 1$, then $\mathrm{V}_{r-1} = \mathrm{V}_r$. As

$$\begin{aligned}(r+s)[\mathrm{V}_0 - \mu_L(\mathscr{E})] &= (s+1)[\mathrm{V}_0 - \mathrm{W}] - \mathrm{V}_{r-1} + [(r-1)\mathrm{V}_0 - (\mathrm{V}_0 + \ldots + \mathrm{V}_{r-2})] \\ &\geq (s+1)\mathrm{V}_r - \mathrm{V}_{r-1} \quad \text{because } \mathrm{V}_0 - \mathrm{W} \geq \mathrm{V}_r \\ &\geq s\mathrm{V}_r\end{aligned}$$

we deduce that $\mathscr{T}_X(-\log D_{v_r})$ is not semistable with respect to L by Proposition 3.2.5.

Let $r \geq 1$. We now assume that $a_{r-1} = 0$ and $a_r = 1$. By using the expressions of Section 2.2.3, we have $\mathrm{V}_0 = \ldots = \mathrm{V}_{r-1} = \mathrm{V}$ where

$$\mathrm{V} = \sum_{k=0}^{s} \binom{s+r-1}{k} \nu^k \quad \text{and} \quad \mathrm{W} = \sum_{k=0}^{s-1} \binom{s+r-1}{k} \nu^k \,.$$

The points 4 and 5 of Proposition 3.2.5 are not verified in this case. To check the stability of $\mathscr{E}$ it is enough to compare

$$\mu_L(\mathscr{E}) = \frac{r\mathrm{V} + (s+1)\mathrm{W}}{r+s}$$

with $\max(\mathrm{V}, \mathrm{W})$. We have $(r+s)(\mu_L(\mathscr{E}) - \mathrm{W}) = r\mathrm{V} - (r-1)\mathrm{W} > 0$ because $\mathrm{W} < \mathrm{V}$ and

$$\begin{aligned}(r+s)(\mu_L(\mathscr{E}) - \mathrm{V}) =&(s+1)\mathrm{W} - s\mathrm{V} \\ =&\sum_{k=0}^{s-1} \binom{s+r-1}{k} \nu^k - s\binom{s+r-1}{s} \nu^s = \mathrm{P}_0(\nu) \,.\end{aligned}$$

By the sign rule of Descartes (see Theorem 3.3.9), the polynomial P_0 has a unique positive root ν_0. If $\nu > 0$, then $\mathrm{P}_0(\nu) > 0$ (resp. $\mathrm{P}_0(\nu) \geq 0$) if and only if $\nu < \nu_0$ (resp. $\nu \leq \nu_0$). Thus, $\mathscr{T}_X(-\log D_{v_r})$ is stable (resp. semistable) with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $0 < \nu < \nu_0$ (resp. $0 < \nu \leq \nu_0$). □

3.3.4. Sum of divisors coming from the base and the bundle: second part. In this part we study the stability of the logarithmic tangent bundle $\mathscr{T}_X(-\log D)$ when $r \geq 2$ and

$$D \in \{D_{v_0}\} \cup \{D_{v_0} + D_{w_j} : 0 \leq j \leq s\} \cup \{D_{v_0} + D_{v_i} : 2 \leq i \leq r\} \,.$$

The last case $D = D_{v_0} + D_{v_1}$ will be studied in Section 3.3.5.

Lemma 3.3.14. *Let $r \geq 2$, $(a_1, \dots, a_r) \neq (0, \dots, 0)$ such that $a_1 < a_r$, $i \in \{2, \dots, r\}$ and $j \in \{0, \dots, s\}$. We set $\mathscr{E} = \mathscr{T}_X(-\log D_{v_0})$, $\mathscr{F} = \mathscr{T}_X(-\log(D_{v_0}+D_{v_i}))$ and $\mathscr{G} = \mathscr{T}_X(-\log(D_{v_0}+D_{w_j}))$. For any $L \in \mathrm{Amp}(X)$, the vector bundles $\mathscr{E}$, $\mathscr{F}$ and $\mathscr{G}$ are not semistable with respect to L.*

Proof. We have $\mu_L(\mathscr{E}) > \mu_L(\mathscr{F})$ and $\mu_L(\mathscr{E}) > \mu_L(\mathscr{G})$. We will show that $\mathrm{V}_1 > \mu_L(\mathscr{E})$. By Lemma 3.2.9, we have $\mathrm{V}_1 - \mathrm{W} \geq \mathrm{V}_r$. Therefore

$$\begin{aligned}(r+s)(\mathrm{V}_1 - \mu_L(\mathscr{E})) &= (r+s)\mathrm{V}_1 - (\mathrm{V}_1 + \ldots + \mathrm{V}_r) - (s+1)\mathrm{W} \\ &= (s+1)(\mathrm{V}_1 - \mathrm{W}) - \mathrm{V}_r + ((r-1)\mathrm{V}_1 - (\mathrm{V}_1 + \ldots + \mathrm{V}_{r-1})) \\ &\geq (s+1)(\mathrm{V}_1 - \mathrm{W}) - \mathrm{V}_r \\ &\geq s\mathrm{V}_r\end{aligned}$$

By Proposition 3.2.5, $\mathscr{E}$, $\mathscr{F}$ and $\mathscr{G}$ are not semistable with respect to L. □

Let $a \in \mathbb{N}^*$. We now study what happens in Lemma 3.3.14 when $a_1 = \ldots = a_r = a$. We first consider the case $r = 1$.

Theorem 3.3.15. *We assume that $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(a))$. Let P_1 and Q be the polynomials defined by*

$$\mathrm{P}_1(x) = (s+1)\sum_{k=0}^{s-1}\binom{s}{k}a^{s-k-1}x^k - s\,x^s \quad \textit{and} \quad \mathrm{Q}(x) = x^s - \sum_{k=0}^{s-1}\binom{s}{k}a^{s-k-1}x^k\,.$$

We have :

1. *$\mathscr{T}_X(-\log D_{v_0})$ is stable (resp. semistable) with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $0 < \nu < \nu_1$ (resp. $0 < \nu \leq \nu_1$) where ν_1 is the unique positive root of P_1.*
2. *If $j \in \{0, \dots, s\}$, then $\mathscr{T}_X(-\log(D_{v_0} + D_{w_j}))$ is semistable with respect to $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $\nu = \nu_2$ where ν_2 is the unique positive root of Q.*
3. *$\varnothing = \mathrm{Stab}(\,\mathscr{T}_X(-\log(D_{v_0} + D_{v_1}))\,) \subsetneq \mathrm{sStab}(\,\mathscr{T}_X(-\log(D_{v_0} + D_{v_1}))\,) = \mathrm{Amp}(X)\,.$*

Proof. By the sign rule of Descartes (Theorem 3.3.9), P_1 and Q have a unique positive root. Let $L = \pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$, by using the expressions of Section 2.2.3, we have :

$$\mathrm{V}_1 = \nu^s \quad \text{and} \quad \mathrm{W} = \sum_{k=0}^{s-1}\binom{s}{k}a^{s-k-1}\nu^k.$$

By Proposition 3.2.5, to check the stability of $\mathscr{E} = \mathscr{T}_X(-\log D_{v_0})$, it is enough to compare

$$\mu_L(\mathscr{E}) = \frac{\mathrm{V}_1 + (1+s)\mathrm{W}}{1+s}$$

with $\max(\mathrm{V}_1, \mathrm{W})$. We have $\mu_L(\mathscr{E}) > \mathrm{W}$ and $(1+s)(\mu_L(\mathscr{E}) - \mathrm{V}_1) = \mathrm{P}_1(\nu)$. Thus, $\mathscr{E}$ is stable (resp. semistable) with respect to L if and only if $0 < \nu < \nu_1$ (resp. $0 < \nu \leq \nu_1$).

Let $\mathscr{F} = \mathscr{T}_X(-\log(D_{v_0} + D_{w_j}))$. By the point 1 of Lemma 3.3.7, $\mathscr{F}$ is semistable with respect to L if and only if $\mathrm{V}_1 = \mathrm{W}$. As $\mathrm{Q}(\nu) = \mathrm{V}_1 - \mathrm{W}$, we deduce that $\mathscr{F}$ is semistable with respect to L if and only if $\nu = \nu_2$.

Let $\mathscr{G} = \mathscr{T}_X(-\log(D_{v_0} + D_{v_1}))$. By the point 3 of Lemma 3.3.7, we have

$$\mathscr{G} \cong \bigoplus_{l=0}^{s}\mathscr{O}_X(D_{w_l}).$$

As for any l, $\mu_L(\mathscr{O}_X(D_{w_l})) = \mathrm{W}$, we deduce that $\mathscr{G}$ is polystable with respect to L. □

We now consider the case $r \geq 2$ and $a_1 = \ldots = a_r = a$ with $a \in \mathbb{N}^*$.

Lemma 3.3.16. *We have* $\operatorname{card}\{(\alpha_1, \ldots, \alpha_p) \in \mathbb{N}^p : \alpha_1 + \ldots + \alpha_p = m\} = \binom{m+p-1}{m}$.

We recall that $V_{1s} = 1$. By Lemma 3.3.16, for all $k \in \{0, \ldots, s-1\}$,

$$W_k = \sum_{d_1+\ldots+d_r=s-k-1} a_1^{d_1} \cdots a_r^{d_r} = \binom{s-k+r-2}{s-k-1} a^{s-k-1}$$

and

$$V_{1k} = \sum_{d_2+\ldots+d_r=s-k} a_2^{d_2} \cdots a_r^{d_r} = \binom{s-k+r-2}{s-k} a^{s-k} \; .$$

By using the equality $\binom{n}{p-1} = \dfrac{p}{n-p+1}\binom{n}{p}$, for any $k \in \{0, \ldots, s-1\}$,

$$W_k = \binom{s-k+r-2}{s-k-1} a^{s-k-1} = \frac{s-k}{r-1}\binom{s-k+r-2}{s-k} a^{s-k-1} = \frac{s-k}{a(r-1)} V_{1k} \; .$$

Theorem 3.3.17. *Let* $r \geq 2$ *and* $X = \mathbb{P}\left(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(a_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^s}(a_r)\right)$ *with* $a_1 = \ldots = a_r = a$ *where* $a \in \mathbb{N}^*$. *We set* $\mathscr{E} = \mathscr{T}_X(-\log D_{v_0})$. *Let* P_1 *and* Q *be the polynomials defined by:*

$$P_1(x) = \sum_{k=0}^{s-1} \left[\left(-s + \frac{(s-k)(s+1)}{a(r-1)}\right) \binom{s+r-1}{k} V_{1k} \right] x^k - s\binom{s+r-1}{s} x^s \, ,$$

$$Q(x) = \sum_{k=0}^{s-1} \left[\left(r - \frac{s-k}{a}\right) \binom{s+r-1}{k} V_{1k} \right] x^k + r\binom{s+r-1}{s} x^s \, .$$

We have :

1. *If* $a < \dfrac{s}{r}$, *then* $\mathscr{E}$ *is stable (resp. semistable) with respect to* $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ *if and only if* $\nu_2 < \nu < \nu_1$ *(resp.* $\nu_2 \leq \nu \leq \nu_1$*) where* ν_1 *and* ν_2 *are respectively the positive roots of* P_1 *and* Q.
2. *If* $\dfrac{s}{r} \leq a < \dfrac{s+1}{r-1}$, *then* $\mathscr{E}$ *is stable (resp. semistable) with respect to* $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ *if and only if* $0 < \nu < \nu_1$ *(resp.* $0 < \nu \leq \nu_1$*) where* ν_1 *is the positive root of* P_1.
3. *If* $a \geq \dfrac{s+1}{r-1}$, *then for any* $L \in \operatorname{Amp}(X)$, $\mathscr{E}$ *is not semistable with respect to* L.

Proof. We first explain the condition which ensures the existence of positive roots on P_1 and Q. We write

$$P_1(x) = \sum_{k=0}^{s} \alpha_k \, x^k \quad \text{and} \quad Q(x) = \sum_{k=0}^{s} \beta_k \, x^k \; .$$

For $k \in \{0, \ldots, s-1\}$, $\alpha_k > 0$ if and only if $k < \left(1 - \dfrac{a(r-1)}{s+1}\right) s$. Therefore,

- If $\dfrac{a(r-1)}{s+1} \geq 1$, then for any $x \geq 0$, $P_1(x) < 0$.
- If $\dfrac{a(r-1)}{s+1} < 1$, then P_1 has only one positive root ν_1.

For $k \in \{0, \ldots, s-1\}$, $\beta_k < 0$ if and only if $k < s - ra$. Therefore,

- If $ra \geq s$, then for any $x \geq 0$, $Q(x) > 0$.
- If $ra < s$, then Q has only one positive root ν_2.

We now show that : If $a < \dfrac{s}{r}$, then $\nu_2 < \nu_1$. As

$$\begin{aligned}\frac{\mathrm{P}_1(x)}{-s} - \frac{\mathrm{Q}(x)}{r} &= \sum_{k=0}^{s-1}\left[\left(1 - \frac{(s-k)(s+1)}{a\,s(r-1)} - 1 + \frac{s-k}{r\,a}\right)\binom{s+r-1}{k}\mathrm{V}_{1k}\right]x^k \\ &= \frac{-(r+s)}{a\,s\,r(r-1)}\sum_{k=0}^{s-1}(s-k)\binom{s+r-1}{k}\mathrm{V}_{1k}\,x^k = \mathrm{P}(x)\end{aligned}$$

and $\dfrac{\mathrm{P}_1(\nu_2)}{-s} - \dfrac{\mathrm{Q}(\nu_2)}{r} = \mathrm{P}(\nu_2) < 0$, we deduce that $\mathrm{P}_1(\nu_2) > 0$. By using the fact that, for $x \geq 0$, $\mathrm{P}_1(x) > 0$ if and only if $0 \leq x < \nu_1$, we deduce that $\nu_2 < \nu_1$.

We can now study the stability of $\mathscr{E}$. As $a_1 = \ldots = a_r$, we have $\mathrm{V}_1 = \ldots = \mathrm{V}_r$. Therefore

$$\mu_L(\mathscr{E}) = \frac{r\mathrm{V}_1 + (s+1)\mathrm{W}}{r+s}\,.$$

By Proposition 3.2.5, to check the stability of $\mathscr{E}$, it is enough to compare $\mu_L(\mathscr{E})$ with $\max(\mathrm{V}_1, \mathrm{W})$. We have

$$\begin{aligned}(r+s)(\mu_L(\mathscr{E}) - \mathrm{V}_1) &= -s\mathrm{V}_1 + (s+1)\mathrm{W} \\ &= -s\sum_{k=0}^{s}\binom{s+r-1}{k}\mathrm{V}_{1k}\,\nu^k + (s+1)\sum_{k=0}^{s-1}\binom{s+r-1}{k}\mathrm{W}_k\,\nu^k \\ &= \mathrm{P}_1(\nu)\end{aligned}$$

and

$$\begin{aligned}(r+s)(\mu_L(\mathscr{E}) - \mathrm{W}) &= r\mathrm{V}_1 - (r-1)\mathrm{W} \\ &= r\sum_{k=0}^{s}\binom{s+r-1}{k}\mathrm{V}_{1k}\,\nu^k - (r-1)\sum_{k=0}^{s-1}\binom{s+r-1}{k}\mathrm{W}_k\,\nu^k \\ &= \mathrm{Q}(\nu)\end{aligned}$$

Therefore,

i. If $a \geq \dfrac{s+1}{r-1}$, then for any $\nu > 0$, we have $\mathrm{P}_1(\nu) < 0$.
ii. If $a < \dfrac{s+1}{r-1}$, then $\mathrm{P}_1(\nu) > 0$ (resp. $\mathrm{P}_1(\nu) \geq 0$) if and only if $0 < \nu < \nu_1$ (resp. $0 < \nu \leq \nu_1$).
iii. If $a \geq \dfrac{s}{r}$, then for any $\nu > 0$, we have $\mathrm{Q}(\nu) > 0$.
iv. If $a < \dfrac{s}{r}$, then $\mathrm{Q}(\nu) > 0$ (resp. $\mathrm{Q}(\nu) \geq 0$) if and only if $\nu > \nu_2$ (resp. $\nu \geq \nu_2$).

The point *i.* shows the third point of the theorem. By using the points *ii.* and *iv.*, we get the first point of theorem. Finally, the points *ii.* and *iii.* give the second point of the theorem. □

We now study the stability of $\mathscr{F}_j = \mathscr{T}_X(-\log(D_{v_0}+D_{w_j}))$ and $\mathscr{G}_i = \mathscr{T}_X(-\log(D_{v_0}+D_{v_i}))$ where $i \in \{1, \ldots, r\}$ and $j \in \{0, \ldots, s\}$.

Proposition 3.3.18. *We assume that* $r \geq 2$ *and* $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^s} \oplus \mathscr{O}_{\mathbb{P}^s}(a_1) \oplus \ldots \oplus \mathscr{O}_{\mathbb{P}^s}(a_r))$ *with* $a_1 = \ldots = a_r = a$ *where* $a \in \mathbb{N}^*$. *Let* $i \in \{1, \ldots, r\}$ *and* $j \in \{0, \ldots, s\}$. *We set* $\mathscr{F}_j = \mathscr{T}_X(-\log(D_{v_0} + D_{w_j}))$, $\mathscr{G}_i = \mathscr{T}_X(-\log(D_{v_0} + D_{v_i}))$ *and*

$$\mathrm{Q}(x) = \sum_{k=0}^{s-1}\left[\left(1 - \frac{s-k}{a(r-1)}\right)\binom{s+r-1}{k}\mathrm{V}_{1k}\right]x^k + \binom{s+r-1}{s}x^s\,.$$

1. *If* $a \geq \dfrac{s}{r-1}$*, then for any* $L \in \mathrm{Amp}(X)$*,* $\mathscr{F}_j$ *and* $\mathscr{G}_i$ *are unstable with respect to* L*.*
2. *If* $a < \dfrac{s}{r-1}$*, then* $\mathscr{F}_j$ *and* $\mathscr{G}_i$ *are polystable with respect to* $\pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ *if and only if* $\nu = \nu_2$ *where* ν_2 *is the unique root of* Q*.*

Proof. We first study the polynomial Q. We write $\mathrm{Q}(x) = \sum_{k=0}^{s} \alpha_k x^k$. For $k \in \{0, \ldots, s-1\}$, $\alpha_k > 0$ if and only if $k < s - a(r-1)$.

- If $a \geq \dfrac{s}{r-1}$, then for any $x \geq 0$, $\mathrm{Q}(x) > 0$.
- If $a < \dfrac{s}{r-1}$, then Q has a unique positive root ν_2.

As $a_1 = \ldots = a_r$, for any $k \in \{1, \ldots, r\}$, $\mu_L(\mathscr{O}_X(D_{v_k})) = \mathrm{V}_1$ and for any $l \in \{0, \ldots, s\}$ $\mu_L(\mathscr{O}_X(D_{w_l})) = \mathrm{W}$. By Lemma 3.3.7 (points 1 and 3), $\mathscr{F}_j$ and $\mathscr{G}_i$ are direct sum of line bundles. Hence, $\mathscr{F}_j$ and $\mathscr{G}_i$ are polystable if and only if $\mathrm{W} = \mathrm{V}_1$. As $\mathrm{Q}(\nu) = \mathrm{V}_1 - \mathrm{W}$, the sheaves $\mathscr{F}_j$ and $\mathscr{G}_i$ are polystable if and only if ν is positive root of Q. □

3.3.5. Sum of divisors coming from the bundle. In this part, we assume that $\mathscr{E} = \mathscr{T}_X(-\log(D_{v_0} + D_{v_1}))$. We will study the stability of $\mathscr{E}$ when $r \geq 2$ and $a_1 < a_r$. The stability of $\mathscr{E}$ when $r = 1$ was treated in Theorem 3.3.15. When $r \geq 2$, in Proposition 3.3.18, we studied the stability of $\mathscr{E}$ when $a_1 = \ldots = a_r$.

Proposition 3.3.19. *Let* $(a_1, \ldots, a_r) \neq (0, \ldots, 0)$ *and* $\mathscr{E} = \mathscr{T}_X(-\log(D_{v_0} + D_{v_1}))$*.*

1. *If* $a_1 = 0$*, then for any* $L \in \mathrm{Amp}(X)$*,* $\mathscr{E}$ *is unstable with respect to* L*.*
2. *If* $r \geq 3$ *and* $a_2 < a_r$*, then for any* $L \in \mathrm{Amp}(X)$*,* $\mathscr{E}$ *is unstable with respect to* L*.*

Proof. We have
$$\mu_L(\mathscr{E}) = \frac{(s+1)\mathrm{W} + \mathrm{V}_2 + \ldots + \mathrm{V}_r}{r+s}.$$

First point. As $\mathrm{card}\{2, \ldots, r\} = r - 1$, by using the point 4 of Proposition 3.2.5 with $I' = \{2, \ldots, r\}$, we get
$$\frac{1}{r+s-1}\left(\sum_{i \in I'} \mathrm{V}_i + (s+1)\mathrm{W}\right) = \frac{1}{r+s-1}\left(\mathrm{V}_2 + \ldots + \mathrm{V}_r + (s+1)\mathrm{W}\right).$$
Thus, $\mathscr{E}$ is not semistable with respect to L.

Second point. We can assume $a_1 \geq 1$. By Lemma 3.3.7 (point 3), $\mathscr{E}$ is a direct sum of line bundles and
$$\mu_L(\mathscr{O}_X(D_{v_2})) = \mathrm{V}_2 > \mathrm{V}_r = \mu_L(\mathscr{O}_X(D_{v_r}))$$
by Lemma 3.2.9. Therefore, $\mathscr{E}$ is not polystable with respect to L. □

We now assume that $0 < a_1 < a_2 = \ldots = a_r$. By Lemma 3.3.7 (point 3), $\mathscr{E}$ is polystable if and only if $\mathrm{V}_2 = \mathrm{W}$.

Proposition 3.3.20. *Let* $r \geq 2$ *and* $(a_1, \ldots, a_r) \neq (0, \ldots, 0)$ *such that* $0 < a_1 < a_2 = \ldots = a_r$*. We denote by* $\ell : \mathbb{N}^* \longrightarrow \mathbb{R}$ *the map defined by*
$$\ell(p) = \sum_{j=0}^{p-1} \binom{j+r-2}{j}\left(1 - \frac{a_r(r-2)}{j+1}\right)\left(\frac{a_r}{a_1}\right)^j - a_1.$$
Then, the logarithmic tangent bundle $\mathscr{T}_X(-\log(D_{v_0} + D_{v_1}))$ *is polystable with respect to* $L = \pi^*\mathscr{O}_{\mathbb{P}^s}(\nu) \otimes \mathscr{O}_X(1)$ *if and only if*
$$\mathrm{Q}(\nu) := \deg_L(\mathscr{O}_X(D_{w_0})) - \deg_L(\mathscr{O}_X(D_{v_2})) = 0.$$
Moreover, there is $\nu_2 > 0$ *such that* $\mathrm{Q}(\nu_2) = 0$ *if and only if* $\ell(s) > 0$*.*

Proof. We will search a condition on a_1, a_2, r and s which ensures the existence of a positive root on Q. We set $a = a_1$ and $b = a_r$. We use the numbers W_k and V_{ik} defined before Lemma 3.2.8. By Lemma 3.3.16, for any $k \in \{0, \dots, s-1\}$, we have

$$\mathrm{W}_k = \sum_{\substack{d_1+\ldots+d_r \\ =s-k-1}} a_1^{d_1} \cdots a_r^{d_r} = \sum_{j=0}^{s-k-1} a^{s-k-1-j} \left(\sum_{d_2+\ldots+d_r=j} b^j \right) = \sum_{j=0}^{s-k-1} \binom{j+r-2}{j} b^j \, a^{s-k-1-j}.$$

We assume that $r \geq 3$. As $\mathrm{V}_2 = \mathrm{V}_r$, then for any $k \in \{0, \dots, s-1\}$,

$$\begin{aligned} \mathrm{V}_{2k} = \sum_{\substack{d_1+\ldots+d_{r-1} \\ =s-k}} a_1^{d_1} \cdots a_{r-1}^{d_{r-1}} &= \sum_{j=0}^{s-k} a^{s-k-j} \left(\sum_{d_2+\ldots+d_{r-1}=j} b^j \right) \\ &= \sum_{j=0}^{s-k} \binom{j+r-3}{j} b^j \, a^{s-k-j} \\ &= a^{s-k} + \sum_{j=0}^{s-k-1} \frac{(r-2)b}{j+1} \binom{j+r-2}{j} b^j \, a^{s-k-1-j}. \end{aligned}$$

The last equality is also true when $r = 2$. For the following, $r \geq 2$. By definition of Q, one has

$$\mathrm{Q} = \sum_{k=0}^{s-1} \binom{s+r-1}{k} (\mathrm{W}_k - \mathrm{V}_{2k}) \nu^k - \binom{s+r-1}{s} \nu^s.$$

As $\mathrm{W}_k - \mathrm{V}_{2k} = a^{s-k-1} \ell(s-k)$, we get

$$\begin{aligned} \mathrm{Q} &= \sum_{k=0}^{s-1} \binom{s+r-1}{k} a^{s-k-1} \ell(s-k) \nu^k - \binom{s+r-1}{s} \nu^s \\ &= -\binom{s+r-1}{s} \nu^s + \sum_{k=1}^{s} \binom{s+r-1}{s-k} a^{k-1} \ell(k) \, \nu^{s-k}. \end{aligned}$$

If $1 \leq k \leq b(r-2)$, then for any $j \in \{0, \dots, k-1\}$,

$$1 - \frac{b(r-2)}{j+1} \leq 1 - \frac{b(r-2)}{k} = \frac{k - b(r-2)}{k} \leq 0.$$

Therefore, $\ell(k) < 0$ if $1 \leq k \leq b(r-2)$. By using the fact that

$$\ell(k+1) = \ell(k) + \left(1 - \frac{b(r-2)}{k+1} \right) \binom{k+r-2}{k} \frac{b^k}{a^k}$$

we deduce that the sequence $(\ell(k))_{k \geq b(r-2)}$ is strictly increasing. Hence,

- If $\ell(s) \leq 0$, then for all $\nu > 0$, we have $\mathrm{Q}(\nu) < 0$;
- If $\ell(s) > 0$, then Q has a unique positive root ν_2.

This completes the proof. □

Lemma 3.3.21. *Let $\ell : \mathbb{N}^* \longrightarrow \mathbb{R}$ be the map given in Proposition 3.3.20.*

1. *If $r = 2$, then $\ell(s) > 0$ if and only if $s > \dfrac{\ln(1 + a_2 - a_1)}{\ln(a_2) - \ln(a_1)}$.*
2. *If $r \geq 3$, then the integer $\delta \in \mathbb{N}$ satisfying $\ell(\delta) \leq 0$ and $\ell(\delta + 1) > 0$ is in the set*

$$\left\{(r-2)a_r, (r-2)a_r + 1, \ldots, \left\lfloor \left(r - 1 + \sqrt{2r-3}\right) a_r + 1 \right\rfloor \right\}$$

where $\lfloor x \rfloor$ is the floor of $x \in \mathbb{R}$.

Proof. If $r = 2$, then

$$\ell(s) = \sum_{j=0}^{s-1} \left(\frac{a_r}{a_1}\right)^j - a_1 = \frac{\left(\frac{a_r}{a_1}\right)^s - 1}{\frac{a_r}{a_1} - 1} - a_1 = \frac{a_1}{a_r - a_1}\left(\left(\frac{a_r}{a_1}\right)^s - (1 + a_r - a_1)\right).$$

Therefore, $\ell(s) > 0$ if and only if $s > \dfrac{\ln(1 + a_r - a_1)}{\ln(a_r) - \ln(a_1)}$.

We now show the second point. We set $m = (r-2)a_r$, $a = a_1$ and $b = a_r$. In the proof of Proposition 3.3.20, we have seen that $\ell(p) < 0$ if $1 \leq p \leq m$ and the sequence $(\ell(p))_{p \geq m}$ is strictly increasing. Hence, the integer δ satisfies $\delta \geq m$. Let $p \geq m$, we have

$$\ell(p) = \sum_{j=0}^{m-1} \binom{j+r-2}{j}\left(1 - \frac{m}{j+1}\right)\left(\frac{b}{a}\right)^j - a + \underbrace{\sum_{j=m}^{p-1} \binom{j+r-2}{j}\left(1 - \frac{m}{j+1}\right)\left(\frac{b}{a}\right)^j}_{:=\beta_p}.$$

We search an integer p such that

$$\beta_p \geq a + \sum_{j=0}^{m-1} \frac{m}{j+1}\binom{j+r-2}{j}\left(\frac{b}{a}\right)^j. \tag{3.9}$$

We have

$$\beta_p = \sum_{k=0}^{p-1-m} \frac{k+1}{k+1+m}\binom{k+m+r-2}{k+m}\left(\frac{b}{a}\right)^{k+m}.$$

By formulas

$$\sum_{j=q}^{n} \binom{j}{q} = \binom{n+1}{q+1} \quad \text{and} \quad \binom{n}{q-1} = \frac{q}{n-(q-1)}\binom{n}{q}$$

for $q \leq n$, we get

$$\begin{aligned}
\binom{k+m+r-2}{r-2} &= \sum_{j=r-3}^{k+m+r-3} \binom{j}{r-3} \\
&= 1 + \sum_{j=0}^{k+m-1} \binom{j+r-2}{r-3} \\
&= 1 + \sum_{j=0}^{k+m-1} \frac{r-2}{j+1}\binom{j+r-2}{j}
\end{aligned}$$

Hence,

$$\beta_p \geq \sum_{k=0}^{p-1-m} \frac{k+1}{k+1+m}\left(\frac{b}{a}\right)^m \left(1 + \sum_{j=0}^{m-1} \frac{r-2}{j+1}\binom{j+r-2}{j}\right).$$

As
$$b\left(\frac{b}{a}\right)^m\left(1+\sum_{j=0}^{m-1}\frac{r-2}{j+1}\binom{j+r-2}{j}\right)>a+\sum_{j=0}^{m-1}\frac{m}{j+1}\binom{j+r-2}{j}\left(\frac{b}{a}\right)^j,$$
we deduce that
$$\beta_p>\frac{1}{b}\sum_{k=0}^{p-1-m}\frac{k+1}{k+1+m}\left(a+\sum_{j=0}^{m-1}\frac{m}{j+1}\binom{j+r-2}{j}\left(\frac{b}{a}\right)^j\right).$$
By using
$$\sum_{k=0}^{p-1-m}\frac{k+1}{k+1+m}\geq\sum_{k=0}^{p-1-m}\frac{k+1}{p}=\frac{(p-m)(p-m+1)}{2p}\geq\frac{(p-m)^2}{2p}$$
and the fact that $(p-m)^2\geq 2bp$ if and only if $p\geq(m+b)+\sqrt{b(2m+b)}$ (because $p\geq m$), we deduce that: if
$$p_0=\left\lfloor(m+b)+\sqrt{b(2m+b)}\right\rfloor+1$$
then β_{p_0} satisfies (3.9) and then $\ell(p_0)>0$. This shows that $\delta\leq p_0$. □

3.4. Application on toric log smooth del Pezzo pairs

A pair (X,D) is a *toric log smooth del Pezzo pair* if X is a smooth toric surface and D an invariant divisor of X such that $-(K_X+D)$ is ample. The goal of this part is to study the stability of the logarithmic tangent bundle $\mathscr{T}_X(-\log D)$ with respect to $-(K_X+D)$ when the pair (X,D) is toric log del Pezzo.

3.4.1. Complete toric surfaces. We assume that $N=M=\mathbb{Z}^2$ and the pairing $\langle\cdot,\cdot\rangle: M\times N\longrightarrow\mathbb{Z}$ is given by
$$\langle m,u\rangle=a_1b_1+a_2b_2$$
for $m=(a_1,a_2)\in M$ and $u=(b_1,b_2)\in N$. Let Σ be a smooth complete fan in $\mathbb{R}^2$ and X the toric surface associated to Σ. We denote by T the torus of X. There is a family of primitive vectors $\{u_i\in N: 0\leq i\leq n-1\}$ with $n\geq 3$ such that

- $\Sigma=\{0\}\cup\{\mathrm{Cone}(u_i): 0\leq i\leq n-1\}\cup\{\mathrm{Cone}(u_i,u_{i+1}): 0\leq i\leq n-1\}$
- $\det(u_i,u_{i+1})=1$

where $u_n=u_0$. For any $i\in\{0,\dots,n-1\}$, we denote by D_i the divisor corresponding to the ray $\mathrm{Cone}(u_i)$ and we set $\gamma_i=\det(u_{i-1},u_{i+1})$. We have
$$u_{i-1}-\gamma_i\,u_i+u_{i+1}=0\,. \tag{3.10}$$
By Lemma 2.1.31 and (2.7), we get
$$\begin{cases} D_i\cdot D_i=-\gamma_i & \\ D_k\cdot D_i=1 & \text{if } k\in\{i-1,i+1\} \\ D_k\cdot D_i=0 & \text{if } k\notin\{i-1,i,i+1\}\end{cases}. \tag{3.11}$$
Let $L=\sum_i a_iD_i$ be an invariant divisor of X. The polytope corresponding to L is given by
$$P=\{m\in\mathbb{Z}^2:\langle m,u_i\rangle\geq -a_i \text{ for } i\in\{0,\dots,n-1\}\} \tag{3.12}$$

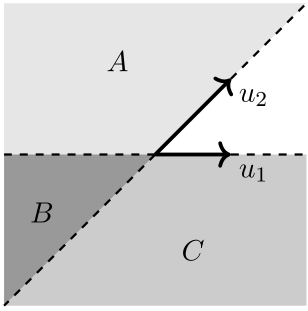


Figure 3.1: Geometry of the fan

and the facet of P corresponding to the vector u_i is given by

$$P_i = \{m \in \mathbb{Z}^2 : \langle m, u_i \rangle = -a_i\} \cap P. \tag{3.13}$$

By the toric Kleiman Criterion (Theorem 2.1.34), L is ample if and only if for any $i \in \{0, \dots, n-1\}$,

$$L \cdot D_i = a_{i+1} + a_{i-1} - \gamma_i a_i > 0. \tag{3.14}$$

Thus, by Proposition 2.1.36, $\mathrm{vol}(P_i) = a_{i+1} + a_{i-1} - \gamma_i a_i$.

Remark 3.4.1. If P is the polytope corresponding to an ample divisor, then the vertices of P are exactly the intersections $P_j \cap P_{j+1}$ for $j \in \{0, \dots, n-1\}$.

3.4.2. Toric log smooth del Pezzo pairs. We use the notations of the previous section. We describe here all toric log smooth del Pezzo pairs. Let X be a toric surface associated to a fan Σ. By Corollary 2.1.17, we have

$$\mathrm{card}(\Sigma(1)) = 2 + \mathrm{rk}(\mathrm{Pic}(X)).$$

Lemma 3.4.2. *Let X be a complete smooth toric surface with Picard rank p and D a reduced invariant divisor of X defined by $D = \sum_{i \in \Delta} D_i$ where $\Delta \subseteq \{0, \dots, n-1\}$.*

1. *If $\mathrm{card}(\Delta) \geq 3$, then $-(K_X + D)$ is not ample.*
2. *If $p \geq 3$ and $\mathrm{card}(\Delta) \in \{1, 2\}$, then $-(K_X + D)$ is not ample.*

Proof. Let $\Delta' = \{0, \dots, n-1\} \setminus \Delta$. By Theorem 2.1.14, we have

$$-(K_X + D) = \sum_{i \in \Delta'} D_i.$$

First point. Let P be the polytope corresponding to $-(K_X + D)$. By (3.13), $0 \in P_i$ for all $i \in \Delta$. Hence, by Remark 3.4.1, we deduce that $-(K_X + D)$ is not ample.

Second point. For the proof of this point, we will use the geometry of the fan (Figure 3.1). Let $A = \{-\alpha u_1 + \beta u_2 : \alpha, \beta \geq 0\}$, $B = \{-\alpha u_1 - \beta u_2 : \alpha, \beta \geq 0\}$ and $C = \{\alpha u_1 - \beta u_2 : \alpha, \beta \geq 0\}$. We start with the case $\mathrm{card}\,\Delta = 1$. We assume that $D = D_1$. We have $-(K_X + D) \cdot D_0 = 1 - \gamma_0$ and $-(K_X + D) \cdot D_2 = 1 - \gamma_2$. If $-(K_X + D)$ is ample, then $\gamma_0 \leq 0$ and $\gamma_2 \leq 0$. As $\gamma_2 = \det(u_1, u_3)$ and $\gamma_0 = \det(u_{n-1}, u_1)$, we deduce that $u_3 \in B \cup C$ and $u_{n-1} \in A$. When $n \geq 5$, this is in contradiction with the fact that if $u_3 \in B \cup C$, then $u_{n-1} \notin A$. Thus, we deduce that $-(K_X + D)$ is not ample.

We now assume that $\mathrm{card}(\Delta) = 2$. After renumbering the indices, we can assume that $D = D_1 + D_j$ with $j \in \{2, \dots, n-1\}$. We first assume that $j \in \{3, \dots, n-1\}$. Let P be the

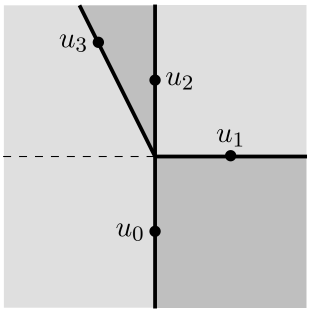


Figure 3.2: Fan of Hirzebruch surface

polytope of $-(K_X + D)$. As $0 \in P_1$ and $0 \in P_j$, we deduce that 0 is a vertex of P. Hence, for any $k \in \{2, \dots, j-1\}$, $\mathrm{vol}(P_k) = 0$. By (3.14), we deduce that $-(K_X + D)$ is not ample.

We now assume that $D = D_1 + D_2$. We have $-(K_X + D) \cdot D_3 = 1 - \gamma_3$ and $-(K_X + D) \cdot D_0 = 1 - \gamma_0$. If $-(K_X + D)$ is ample, then $\gamma_3 \le 0$ and $\gamma_0 \le 0$. As $\gamma_3 = \det(u_2, u_4)$ and $\gamma_0 = \det(u_{n-1}, u_1)$, we deduce that $u_4 \in C$ and $u_{n-1} \in A$. If $n \ge 6$, this situation contradicts the positioning order of vectors u_i. If $n = 5$, we have $u_4 \in A$ and $u_4 \in C$, this is not possible. Therefore, we deduce that $-(K_X + D)$ is not ample. □

If $\Delta \neq \varnothing$, according to Lemma 3.4.2, it is enough to study the positivity of $-(K_X + D)$ when $\mathrm{rk}\,\mathrm{Pic}(X) \in \{1, 2\}$ and $\mathrm{card}(\Delta) \in \{1, 2\}$. Let (e_1, e_2) be the standard basis of $\mathbb{Z}^2$. The rays of the fan of $\mathbb{P}^2$ are the half-line generated by $u_1 = e_1$, $u_2 = e_2$ and $u_0 = -(e_1 + e_2)$.

Proposition 3.4.3. *If $X = \mathbb{P}^2$, then the log smooth pair (X, D) is toric log del Pezzo if and only if $D \in \{D_0, D_1, D_2\} \cup \{D_0 + D_1, D_0 + D_2, D_1 + D_2\}$.*

Proof. We have the linear equivalence $D_1 \sim_{\mathrm{lin}} D_0$ and $D_2 \sim_{\mathrm{lin}} D_0$. By Theorem 2.1.14, we have $K_X = -(D_0 + D_1 + D_2)$, i.e $-K_X \sim_{\mathrm{lin}} 3D_0$. As D_0 is ample, we deduce that $-(K_X + D)$ is not ample if and only if $D = D_1 + D_2 + D_3$. □

We now assume that $X = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r)\right)$ with $r \in \mathbb{N}$. The rays of the fan of X are the half lines generated by the vectors $u_1 = e_1$, $u_2 = e_2$, $u_3 = -e_1 + r\, e_2$ and $u_0 = -e_2$ (Figure 3.2). The numbers γ_i are given by $\gamma_0 = -r$, $\gamma_1 = 0$, $\gamma_2 = r$ and $\gamma_3 = 0$. By (3.14), the divisor $L = a_0\, D_0 + a_1\, D_1 + a_2\, D_2 + a_3\, D_3$ is ample if and only if

$$a_0 + a_2 > 0\ ,\ a_1 + a_3 > r\, a_2\ ,\ a_1 + a_3 > -r\, a_0$$

if and only if

$$a_0 + a_2 > 0 \quad \text{and} \quad a_1 + a_3 > r\, a_2\ . \tag{3.15}$$

We have the linear equivalence of divisors

$$D_1 \sim_{\mathrm{lin}} D_3 \quad \text{and} \quad D_2 \sim_{\mathrm{lin}} D_0 - r\, D_3\ . \tag{3.16}$$

Proposition 3.4.4. *Let $X = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r)\right)$ with $r \in \mathbb{N}$. Then :*

1. *$-K_X$ or $-(K_X + D_0)$ are ample if and only if $r \in \{0,\ 1\}$.*
2. *If $D \in \{D_1,\ D_3,\ D_0 + D_1,\ D_0 + D_3\}$, $-(K_X + D)$ is ample if and only if $r = 0$.*
3. *If $D \in \{D_2,\ D_2 + D_1,\ D_2 + D_3\}$, $-(K_X + D)$ is ample for any $r \in \mathbb{N}$.*
4. *If $D \in \{D_0 + D_2,\ D_1 + D_3\}$, $-(K_X + D)$ is not ample for any $r \in \mathbb{N}$.*

Proof. As $-K_X = D_0 + D_1 + D_2 + D_3$, by (3.16), we have

$$-K_X \sim_{\text{lin}} 2D_0 + (2-r)D_3 \qquad -(K_X + D_0 + D_2) \sim_{\text{lin}} 2D_3$$
$$-(K_X + D_0) \sim_{\text{lin}} D_0 + (2-r)D_3 \qquad -(K_X + D_0 + D_3) \sim_{\text{lin}} D_0 + (1-r)D_3$$
$$-(K_X + D_2) \sim_{\text{lin}} D_0 + 2D_3 \qquad -(K_X + D_2 + D_3) \sim_{\text{lin}} D_0 + D_3$$
$$-(K_X + D_3) \sim_{\text{lin}} 2D_0 + (1-r)D_3 \qquad -(K_X + D_1 + D_3) \sim_{\text{lin}} 2D_0 - rD_3$$

If $a_1 = a_2 = 0$, the condition (3.15) becomes $a_0 > 0$ and $a_3 > 0$. This allows us to conclude. □

3.4.3. Stability with respect to the anti-canonical divisor of the pair. According to Section 3.4.2, we study in this part the stability of the logarithmic tangent bundle $\mathscr{T}_X(-\log D)$ with respect to $-(K_X + D)$ when $-(K_X + D)$ is ample.

Proposition 3.4.5. *Let $X = \mathbb{P}^2$ and D_0, D_1, D_2 be the irreducible invariant divisors of $\mathbb{P}^2$ as in Proposition 3.4.3.*

1. *If $D \in \{D_0, D_1, D_3\}$, then $\mathscr{T}_X(-\log D)$ is polystable but not stable with respect to $\mathscr{O}_{\mathbb{P}^2}(1)$.*
2. *If $D \in \{D_0+D_1, D_0+D_2, D_1+D_2\}$, then $\mathscr{T}_X(-\log D)$ is unstable with respect to $\mathscr{O}_{\mathbb{P}^2}(1)$.*

Proof. The second point follows from Corollary 3.1.13 and the first point follows from Corollary 3.2.2. □

We now consider the case where X is a toric surface of Picard rank two. Let D_0, D_1, D_2, D_3 be the irreducible invariant divisors of X as in Proposition 3.4.4. The divisors D_0, D_1, D_2, D_3 of X defined there are given in Section 2.2.2 by

$$D_0 = D_{v_0} \qquad D_1 = D_{w_1} \qquad D_2 = D_{v_1} \qquad D_3 = D_{w_0}$$

where $v_1 = e_2$ and $w_1 = e_1$.

Proposition 3.4.6. *Let $r \in \{0, 1\}$ and $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(r))$.*

1. *If $r = 0$ and $D \in \{D_i : 0 \leq i \leq 3\} \cup \{D_0 + D_1, D_0 + D_3\} \cup \{D_2 + D_1, D_2 + D_3\}$, then $\mathscr{T}_X(-\log D)$ is polystable with respect to $-(K_X + D)$.*
2. *If $r = 1$, then $\mathscr{T}_X(-\log D_0)$ is stable with respect to $-(K_X + D_0)$.*

Proof. The first point follows from Remark 3.3.4. Let $r = 1$. We have $-(K_X + D_0) \sim_{\text{lin}} D_0 + D_3$ and $\nu = 1$. The polynomial P_1 defined in Theorem 3.3.15 is $\mathrm{P}_1 = 2 - x$. As $0 < \nu < 2$, we deduce that $\mathscr{T}_X(-\log D_0)$ is stable with respect to $-(K_X + D_0)$. □

Proposition 3.4.7. *Let $r \in \mathbb{N}^*$ and $X = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(r))$. If $D \in \{D_2, D_2 + D_1, D_2 + D_3\}$, then $\mathscr{T}_X(-\log D)$ is not semistable with respect to $-(K_X + D)$.*

Proof. If $D \in \{D_2 + D_1, D_2 + D_3\}$, the result follows from Proposition 3.3.11. By Theorem 3.3.13, if $r \geq 2$, then $\mathscr{T}_X(-\log D_2)$ is not semistable with respect to any polarization. If $r = 1$, the polynomial P_0 of Theorem 3.3.13 is given by $\mathrm{P}_0 = 1 - x$. As $-(K_X + D_2) \sim_{\text{lin}} 2D_3 + D_0$ and $\nu = 2$, we deduce that $\mathscr{T}_X(-\log D_2)$ is not semistable with respect to $-(K_X + D_2)$. □

4

TORIC SHEAVES, STABILITY AND FIBRATIONS

For an equivariant reflexive sheaf over a polarised toric variety, we study slope stability of its reflexive pullback along a toric fibration. We show that stability (resp. unstability) is preserved under such pullbacks for so-called adiabatic polarisations. In the strictly semistable situation, under local freeness assumptions, we provide a necessary and sufficient condition on the graded object to ensure stability of the pulled back sheaf. As applications, we provide various stable perturbations of semistable tangent sheaves, either by changing the polarisation, or by blowing-up a subvariety.

4.1. Pullbacks of reflexive sheaves along toric fibrations

Let N and N' be two lattices having respectively M and M' for dual lattices. Let Σ be a complete fan in $N_{\mathbb{R}}$ and Σ' a complete fan in $N'_{\mathbb{R}}$. We denote by X (resp. X') the toric variety associated to the fan Σ (resp. Σ') and T (resp. T') its torus. Given a surjective $\mathbb{Z}$-linear map $\phi : N' \longrightarrow N$ compatible with Σ' and Σ, we denote by $\pi : X' \longrightarrow X$ the induced toric fibration (cf. Section 2.1.2). For an equivariant reflexive sheaf $\mathscr{E}$ on X we set $\mathscr{E}' = (\pi^*\mathscr{E})^{\vee\vee}$ its reflexive pullback on X'.

4.1.1. Pulling back sheaves on a fibration. We first describe part of the family of filtrations of the pulled back sheaves.

Proposition 4.1.1. *Let $\mathscr{E}$ be an equivariant reflexive sheaf on X given by the family of filtrations $\left(E, \{E^\rho(j)\}_{\rho\in\Sigma(1), j\in\mathbb{Z}}\right)$. Let $\left(\widetilde{E}, \{\widetilde{E}^{\rho'}(j)\}_{\rho'\in\Sigma'(1), j\in\mathbb{Z}}\right)$ be the family of filtrations of the equivariant sheaf $(\pi^*\mathscr{E})^{\vee\vee}$. Then we have:*

1. *$\widetilde{E} = E$.*
2. *If $\phi_{\mathbb{R}}(\rho') = \{0\}$, then $\widetilde{E}^{\rho'}(j) = \begin{cases} \{0\} & \text{if } j < 0 \\ E & \text{if } j \geq 0 \end{cases}$.*
3. *If $\phi_{\mathbb{R}}(\rho') = \rho \in \Sigma(1)$ and $\phi(u_{\rho'}) = b_\rho\, u_\rho$, then $\widetilde{E}^{\rho'}(j) = E^\rho\left(\left\lfloor \frac{j}{b_\rho} \right\rfloor\right)$.*

Proof. For $\sigma \in \Sigma$, we define $U_\sigma = \mathrm{Spm}(\mathbb{C}[S_\sigma])$ as an affine open subset of X and for $\sigma' \in \Sigma'$, we define $U'_{\sigma'} = \mathrm{Spm}(\mathbb{C}[S_{\sigma'}])$ as an affine open subset of X'. The sheaf $\pi^*\mathscr{E}$ is defined by

$$\pi^*\mathscr{E} = \pi^{-1}\mathscr{E} \otimes_{\pi^{-1}\mathscr{O}_X} \mathscr{O}_{X'} \tag{4.1}$$

where for any sheaf $\mathscr{F}$ on X, $\pi^{-1}\mathscr{F}$ is defined by

$$\Gamma(U', \pi^{-1}\mathscr{F}) = \varinjlim_{U \supseteq \phi(U')} \Gamma(U, \mathscr{F}) .$$

We have $\Gamma(T, \mathscr{E}) = E \otimes_{\mathbb{C}} \mathbb{C}[M]$. As $\pi(T') = T$, we deduce that

$$\Gamma(T', \mathscr{E}') = (E \otimes_{\mathbb{C}} \mathbb{C}[M]) \otimes_{\mathbb{C}[M]} \mathbb{C}[M'] \cong E \otimes_{\mathbb{C}} \mathbb{C}[M'] \,.$$

Thus, $\widetilde{E} = E$.

Let $\rho' \in \Sigma'(1)$ such that $\phi_{\mathbb{R}}(\rho') = \{0\}$. By Lemma 2.1.21 we get $\pi(O(\rho')) = T$. Hence, by the Orbit-Cone Correspondence, $\pi(U_{\rho'}) = T$. By (4.1), we deduce that

$$\widetilde{E}^{\rho'} = \Gamma(U'_{\rho'}, \mathscr{E}') = (E \otimes_{\mathbb{C}} \mathbb{C}[M]) \otimes_{\mathbb{C}[M]} \mathbb{C}[S_{\rho'}] = E \otimes_{\mathbb{C}} \mathbb{C}[S_{\rho'}] \,.$$

If $m \in S_{\rho'}$, then $\widetilde{E}^{\rho'}_m = E$, otherwise $\widetilde{E}^{\rho'}_m = \{0\}$. This is equivalent to the assumption that $\widetilde{E}^{\rho}(j) = E$ if $j \geq 0$ and $\widetilde{E}^{\rho}(j) = \{0\}$ if $j < 0$.

We now consider the case where $\phi_{\mathbb{R}}(\rho') = \rho$ and $\phi(u_{\rho'}) = b_\rho\, u_\rho$. By Lemma 2.1.21, we have $\pi(O(\rho')) = O(\rho)$ and then $\pi(U'_{\rho'}) = U_\rho$. Hence,

$$\widetilde{E}^{\rho'} = E^{\rho} \otimes_{\mathbb{C}[S_\rho]} \mathbb{C}[S_{\rho'}].$$

As $\phi : N' \longrightarrow N$ is surjective, there is an injective map $\psi : M \longrightarrow M'$ such that for any $m \in M$ and $u' \in N'$, $\langle m, \phi(u')\rangle = \langle \psi(m), u'\rangle$. Let $e_\rho \in M$ such that $M_{\mathbb{R}} = \mathbb{R}e_\rho \oplus \mathrm{Span}(u_\rho)^{\perp}$ with $\langle e_\rho, u_\rho\rangle = 1$ and let $e_{\rho'} \in M'$ such that $M'_{\mathbb{R}} = \mathbb{R}e_{\rho'} \oplus \mathrm{Span}(u_{\rho'})^{\perp}$ with $\langle e_{\rho'}, u_{\rho'}\rangle = 1$. We set $M_0 = \mathrm{Span}(u_{\rho'})^{\perp} \cap M'$. There is $m_\rho \in M_0$ such that

$$\psi(e_\rho) = b_\rho\, e_{\rho'} + m_\rho \,.$$

For any $m' \in M'$, there is $a \in \mathbb{Z}$ and $m_0 \in M_0$ such that $m' = a e_{\rho'} + m_0$. Let $(a', r) \in \mathbb{Z}^2$ such that $a = a' b_\rho + r$ with $0 \leq r < b_\rho$, we have

$$m' = r e_{\rho'} + a'(b_\rho e_{\rho'} + m_\rho) + (m_0 - a' m_\rho) = r e_{\rho'} + \psi(a' e_\rho) + (m_0 - a' m_\rho) \,.$$

Thus, $M' = A + \psi(M) + M_0$ where $A = \{k\, e_{\rho'} : 0 \leq k \leq b_\rho - 1\}$. Therefore, $S_{\rho'} = A + \psi(S_\rho) + M_0$ and

$$\mathbb{C}[S_{\rho'}] \cong \bigoplus_{m'' \in A} \mathbb{C}[S_\rho] \otimes_{\mathbb{C}} (\chi^{m''} \cdot \mathbb{C}[M_0])$$

where for $m \in S_\rho$ and $m_0 \in M_0$, $\chi^m \otimes (\chi^{m''} \cdot \chi^{m_0}) = \chi^{m'' + \psi(m) + m_0}$. Thus,

$$\begin{aligned} \widetilde{E}^{\rho'} &\cong \bigoplus_{m'' \in A} E^{\rho} \otimes_{\mathbb{C}} (\chi^{m''} \cdot \mathbb{C}[M_0]) \\ &= \bigoplus_{m'' \in A} \left(\sum_{m \in M,\, m_0 \in M_0} E^{\rho}(\langle m,\, u_\rho\rangle) \otimes \chi^{m'' + \psi(m) + m_0} \right) \end{aligned}$$

As $\langle m, u_\rho\rangle \in \mathbb{Z}$, for any $(m'', m_0) \in A \times M_0$,

$$\langle m, u_\rho\rangle = \frac{\langle \psi(m), u_{\rho'}\rangle}{b_\rho} = \frac{\langle \psi(m) + m_0, u_{\rho'}\rangle}{b_\rho} = \left\lfloor \frac{\langle m'' + \psi(m) + m_0, u_{\rho'}\rangle}{b_\rho} \right\rfloor \,.$$

Thus,

$$\begin{aligned} \widetilde{E}^{\rho'} &\cong \sum_{\substack{m \in M,\, m_0 \in M_0, \\ m'' \in A}} E^{\rho}\left(\left\lfloor \frac{\langle m'' + \psi(m) + m_0, u_{\rho'}\rangle}{b_\rho} \right\rfloor \right) \otimes \chi^{m'' + \psi(m) + m_0} \\ &\cong \sum_{m' \in M'} E^{\rho}\left(\left\lfloor \frac{\langle m', u_{\rho'}\rangle}{b_\rho} \right\rfloor \right) \otimes \chi^{m'} \end{aligned}$$

that is $\widetilde{E}^{\rho'}(j) = E^{\rho}\left(\left\lfloor \frac{j}{b_\rho} \right\rfloor \right)$. □

Notation 4.1.2. Let F be a vector subspace of E. We denote by $\left(F, \{\widetilde{F}^{\rho'}(j)\}\right)$ the family of filtrations of $(\pi^*\mathscr{E}_F)^{\vee\vee}$.

Corollary 4.1.3. *Let F be a vector subspace of E. Then the family of filtrations $\left(F, \{\widetilde{F}^{\rho'}(j)\}\right)$ satisfies $\widetilde{F}^{\rho'}(j) = F \cap \widetilde{E}^{\rho'}(j)$ for all rays ρ' such that $\phi_{\mathbb{R}}(\rho') \in \{0\} \cup \Sigma(1)$.*

Proof. If $\phi_{\mathbb{R}}(\rho') = \{0\}$, we have $\widetilde{F}^{\rho'}(j) = \begin{cases} \{0\} & \text{if } j < 0 \\ F & \text{if } j \geq 0 \end{cases}$; so $\widetilde{F}^{\rho'}(j) = F \cap \widetilde{E}^{\rho'}(j)$.
If $\phi_{\mathbb{R}}(\rho') = \rho \in \Sigma(1)$ and $\phi(u_{\rho'}) = b_\rho\, u_\rho$, we have

$$\widetilde{F}^{\rho'}(j) = F^{\rho}\left(\left\lfloor \frac{j}{b_\rho} \right\rfloor\right) = F \cap E^{\rho}\left(\left\lfloor \frac{j}{b_\rho} \right\rfloor\right) = F \cap \widetilde{E}^{\rho'}(j)\,.$$

□

4.1.2. Slopes of the pulled back sheaves. We now assume that $\pi : X' \longrightarrow X$ is a toric fibration between two complete and $\mathbb{Q}$-factorial toric varieties. We set $n = \dim X$ and $r = \dim X' - \dim X$. Let L be an ample divisor on X and L' a π-ample divisor on X'. For $\varepsilon \in \mathbb{Q}_{>0}$ small enough, $L_\varepsilon = \pi^*L + \varepsilon L' \in \mathrm{Pic}(X') \otimes_{\mathbb{Z}} \mathbb{Q}$ defines an ample $\mathbb{Q}$-divisor on X'. In this section, we relate the slopes of sheaves on X with respect to L to the slopes of their pullbacks on X' with respect to L_ε. All intersection products are made in the Chow rings $A^\bullet(X)_{\mathbb{Q}}$ and $A^\bullet(X')_{\mathbb{Q}}$ (cf. Section 2.1.4).

Proposition 4.1.4. *Let $\mathscr{E}$ be an equivariant reflexive sheaf on X with family of filtrations given by $(E, \{E^\rho(j)\})$ and $\mathscr{E}' = (\pi^*\mathscr{E})^{\vee\vee}$. Then, there is $C > 0$ such that*

$$\mu_{L_\varepsilon}(\mathscr{E}') = C\,\mu_L(\mathscr{E})\,\varepsilon^r - \frac{\varepsilon^{r+1}}{\mathrm{rk}\,\mathscr{E}} \sum_{k=0}^{n-2} \binom{n+r-1}{k} \times \\ \sum_{\rho\in\Sigma(1)} \iota_\rho(\mathscr{E})\,\varepsilon^{n-k-2} (\pi^* D_\rho)\cdot(\pi^*L)^k \cdot (L')^{n+r-k-1} \quad (4.2)$$

and for any vector subspace F of E,

$$\mu_{L_\varepsilon}(\mathscr{E}'_F) = \mu_{L_\varepsilon}((\pi^*\mathscr{E}_F)^{\vee\vee}) - \frac{\varepsilon^{r+1}}{\mathrm{rk}\,F} \sum_{k=0}^{n-2} \binom{n+r-1}{k} \times \\ \sum_{\rho'\in\Delta} \left(\iota_{\rho'}(\mathscr{E}'_F) - \iota_{\rho'}((\pi^*\mathscr{E}_F)^{\vee\vee})\right) \varepsilon^{n-k-2} D_{\rho'}\cdot(\pi^*L)^k \cdot (L')^{n+r-k-1} \quad (4.3)$$

where $\Delta = \{\rho' \in \Sigma'(1) : \phi_{\mathbb{R}}(\rho') \notin \Sigma(0) \cup \Sigma(1)\}$ and $\iota_\rho(\mathscr{E})$ given in (2.19).

Remark 4.1.5. The set Δ indexes the invariant divisors of X' contracted by π.

Proof. First, we have

$$L_\varepsilon^{n+r-1} = \sum_{k=0}^{n+r-1} \binom{n+r-1}{k} (\pi^*L)^k \cdot (\varepsilon L')^{n+r-k-1}.$$

Let D be a divisor on X. By the Projection formula (2.6), for any $k \in \{0, \ldots, n+r-1\}$,

$$\pi_*\left((\pi^*D)\cdot(\pi^*L)^k \cdot (L')^{n+r-k-1}\right) = D \cdot L^k \cdot \pi_*((L')^{n+r-k-1}) \in A_0(X).$$

Hence,

$$\deg\left((\pi^*D)\cdot(\pi^*L)^k\cdot(L')^{n+r-k-1}\right) = \deg\left(D\cdot L^k\cdot\pi_*((L')^{n+r-k-1})\right)\,.$$

If $k \geq n$, then $D\cdot L^k\cdot\pi_*((L')^{n+r-k-1}) = 0 \in A_0(X)$. Hence,

$$(\pi^*D)\cdot(\pi^*L)^k\cdot(L')^{n+r-k-1} = 0.$$

Since L' is relatively ample, if $V \subseteq X'$ is an irreducible subvariety of positive dimension that maps to a point in X, then $(L')^{\dim V}\cdot V > 0$ (cf. [26, Corollay 1.7.9]). So in the case where $k = n-1$, one has

$$(\pi^*D)\cdot(\pi^*L)^{n-1}\cdot(L')^r > 0\,.$$

As $\pi_*((L')^r) \in A_n(X)$, we deduce that there is a constant $C > 0$ such that $\pi_*((L')^r) = C\cdot[X]$. Thus,

$$(\pi^*D)\cdot(\pi^*L)^{n-1}\cdot(L')^r = C\,(D\cdot L^{n-1}) = C\,\deg_L(D)\,.$$

Therefore, the degree of π^*D with respect to L_ε is given by

$$\begin{aligned}\deg_{L_\varepsilon}(\pi^*D) =& C\varepsilon^r\,\deg_L(D)\\ &+\varepsilon^{r+1}\sum_{k=0}^{n-2}\binom{n+r-1}{k}\varepsilon^{n-k-2}(\pi^*D)\cdot(\pi^*L)^k\cdot(L')^{n+r-k-1}\,.\end{aligned}$$

As $\mu_{L_\varepsilon}(\mathscr{E}') = c_1(\mathscr{E}')\cdot L_\varepsilon^{n+r-1} = \pi^*(c_1(\mathscr{E}))\cdot L_\varepsilon^{n+r-1}$, according to Equations (2.19) and (2.20), we get Formula (4.2).

Let now F be a vector subspace of E. We recall that $\mathscr{E}'_F$ is the saturated subsheaf of $\mathscr{E}'$ associated to F (cf. Notation 2.3.18). We wish to compare the slopes of $\mathscr{E}'_F$ and of $(\pi^*\mathscr{E}_F)^{\vee\vee}$. We denote by $\left(F,\{\widetilde{F}^{\rho'}(j)\}\right)$ the family of filtrations of $(\pi^*\mathscr{E}_F)^{\vee\vee}$. By Corollary 4.1.3, for any $\rho' \in \Sigma'(1)$ such that $\phi_{\mathbb{R}}(\rho') \in \{0\}\cup\Sigma(1)$, $\widetilde{F}^{\rho'}(j) = \widetilde{E}^{\rho'}(j)\cap F$. Therefore, according to (2.19), one has

$$c_1(\mathscr{E}'_F) = c_1((\pi^*\mathscr{E}_F)^{\vee\vee}) - \sum_{\rho'\in\Delta}\left(\iota_{\rho'}(\mathscr{E}'_F) - \iota_{\rho'}((\pi^*\mathscr{E}_F)^{\vee\vee})\right)D_{\rho'}\,. \tag{4.4}$$

Let $\rho' \in \Delta$, then $\dim\pi(D_{\rho'}) \leq n-2$. For $k \in \{0,\dots,n+r-1\}$,

$$D_{\rho'}\cdot(L')^{n+r-k-1} \in A_k(|D_{\rho'}|\cap|(L')^{n+r-k-1}|)$$

and

$$\pi_*\left(D_{\rho'}\cdot(L')^{n+r-k-1}\right) \in A_k\left(\pi(|D_{\rho'}|\cap|(L')^{n+r-k-1}|)\right)\,.$$

As $\dim\pi(|D_{\rho'}|\cap|(L')^{n+r-k-1}|) \leq n-2$, we deduce that $\pi_*\left(D_{\rho'}\cdot(L')^{n+r-k-1}\right) = 0$ if $k \geq n-1$. Thus,

$$(\pi^*L)^k\cdot(\varepsilon L')^{n+r-k-1}\cdot D_{\rho'} = 0$$

if $k \geq n-1$. Therefore, for any $\rho' \in \Delta$,

$$\deg_{L_\varepsilon}(D_{\rho'}) = \varepsilon^{r+1}\sum_{k=0}^{n-2}\binom{n+r-1}{k}\varepsilon^{n-k-2}D_{\rho'}\cdot(\pi^*L)^k\cdot(L')^{n+r-k-1}.$$

Using (4.4) and (2.20), we get Formula (4.3). □

4.1.3. Stability of the pulled back sheaf along a fibration. We now give the main results of this chapter about stability of pulled back sheaves along fibrations. We keep the notations of the previous section. Recall that to check slope stability of $\mathscr{E}'$, by Proposition 2.3.16 and Lemma 2.3.17, it is enough to compare slopes with subsheaves of the form $\mathscr{E}'_F$. According to Formulas (4.2) and (4.3), for any vector subspace F of E, we have

$$\mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}(\mathscr{E}'_F) = \left(\mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}((\pi^*\mathscr{E}_F)^{\vee\vee})\right) + \left(\mu_{L_\varepsilon}((\pi^*\mathscr{E}_F)^{\vee\vee}) - \mu_{L_\varepsilon}(\mathscr{E}'_F)\right) \tag{4.5}$$

where

$$\begin{cases} \mu_{L_\varepsilon}((\pi^*\mathscr{E}_F)^{\vee\vee}) - \mu_{L_\varepsilon}(\mathscr{E}'_F) = o(\varepsilon^r) \\ \mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}((\pi^*\mathscr{E}_F)^{\vee\vee}) = C(\mu_L(\mathscr{E}) - \mu_L(\mathscr{E}_F))\varepsilon^r + o(\varepsilon^r) \end{cases} .$$

As $\mathscr{E}'_F$ is the saturation of $(\pi^*\mathscr{E}_F)^{\vee\vee}$, we have $\mu_{L_\varepsilon}((\pi^*\mathscr{E}_F)^{\vee\vee}) \leq \mu_{L_\varepsilon}(\mathscr{E}'_F)$.

Theorem 4.1.6. *Let $\mathscr{E}$ be a T-equivariant stable reflexive sheaf on (X, L). Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, the reflexive pullback $\mathscr{E}' = (\pi^*\mathscr{E})^{\vee\vee}$ is stable on (X', L_ε).*

Proof. For any vector subspace F of E, we have $\mu_L(\mathscr{E}) - \mu_L(\mathscr{E}_F) > 0$. We set

$$a_0 = \min\{\mu_L(\mathscr{E}) - \mu_L(\mathscr{E}_F) : \{0\} \subsetneq F \subsetneq E\} .$$

By Lemma 2.3.19, one has $a_0 > 0$. As the set $\{\mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}(\mathscr{E}'_F) : \{0\} \subsetneq F \subsetneq E\}$ is finite, we deduce that the number of vector spaces F to consider is finite. By Equation (4.5), we get

$$\mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}(\mathscr{E}'_F) \geq C a_0 \,\varepsilon^r + o(\varepsilon^r) .$$

Thus, there is $\varepsilon_0 > 0$, such that for any $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, $\mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}(\mathscr{E}'_F) > 0$. Hence, we deduce that $\mathscr{E}'$ is stable with respect to L_ε. □

Proposition 4.1.7. *Let $\mathscr{E}$ be a T-equivariant unstable reflexive sheaf on (X, L). Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, the reflexive pullback $\mathscr{E}' = (\pi^*\mathscr{E})^{\vee\vee}$ is unstable on (X', L_ε).*

Proof. There is a vector subspace F of E with $0 < \dim F < \dim E$ such that $\mu_L(\mathscr{E}) - \mu_L(\mathscr{E}_F) < 0$. By (4.5), there is $\varepsilon_0 > 0$, such that for any $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, $\mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}(\mathscr{E}'_F) < 0$. Hence, $\mathscr{E}'$ is unstable with respect to L_ε. □

Remark 4.1.8. Theorem 4.1.6 and Proposition 4.1.7 also follow from the openness property of stability [10, Theorem 3.3].

Our main result deals with the more delicate strictly semistable situation. Let $\mathscr{E}$ be a strictly semistable torsion-free sheaf on (X, L). It then admits a Jordan-Hölder filtration

$$0 = \mathscr{E}_1 \subseteq \mathscr{E}_2 \subseteq \ldots \subseteq \mathscr{E}_\ell = \mathscr{E}$$

by slope semistable coherent subsheaves with stable quotients of the same slope as $\mathscr{E}$ [16]. The reflexive pullbacks of the $\mathscr{E}_i$'s form natural candidates to test for stability of the reflexive pullback of $\mathscr{E}$ on (X', L_ε). In fact, we will see shortly that if $\mathscr{E}$ and

$$\mathrm{Gr}_L(\mathscr{E}) := \bigoplus_{i=1}^{\ell-1} \mathscr{E}_{i+1}/\mathscr{E}_i$$

are locally free, it is actually enough to compare slopes with these sheaves. In order to state our result, we will introduce some notations. Let $\mathfrak{E}$ be the set of equivariant saturated subsheaves of $\mathscr{E}$ arising in a Jordan-Holder filtration for $\mathscr{E}$. For two coherent sheaves $\mathscr{F}_1$ and $\mathscr{F}_2$ on X', we will write $\mu_0(\mathscr{F}_1) < \mu_0(\mathscr{F}_2)$ (resp. $\mu_0(\mathscr{F}_1) \leq \mu_0(\mathscr{F}_2)$ or $\mu_0(\mathscr{F}_1) = \mu_0(\mathscr{F}_2)$) when the coefficient of the smallest exponent in the expansion in ε of $\mu_{L_\varepsilon}(\mathscr{F}_2) - \mu_{L_\varepsilon}(\mathscr{F}_1)$ is strictly positive (resp. greater or equal to zero or equal to zero). We say that a locally free semistable sheaf is called *sufficiently smooth* if its graded object is locally free.

Theorem 4.1.9. *Let $\mathscr{E}$ be a T-equivariant locally free and sufficiently smooth strictly semistable sheaf on (X, L). Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[\cap\mathbb{Q}$, the reflexive pullback $\mathscr{E}' := (\pi^*\mathscr{E})^{\vee\vee}$ on (X', L_ε) is:*

1. *stable iff for all $\mathscr{F} \in \mathfrak{E}$, $\mu_0(\pi^*\mathscr{F}) < \mu_0(\mathscr{E}')$,*
2. *strictly semistable iff for all $\mathscr{F} \in \mathfrak{E}$, $\mu_0(\pi^*\mathscr{F}) \leq \mu_0(\mathscr{E}')$ with at least one equality,*
3. *unstable iff there is one $\mathscr{F} \in \mathfrak{E}$ with $\mu_0(\pi^*\mathscr{F}) > \mu_0(\mathscr{E}')$.*

Proof. Let $\mathfrak{F} = \{F \subsetneq E : \mu_L(\mathscr{E}_F) < \mu_L(\mathscr{E})\}$. By Equation (4.5), for any $F \in \mathfrak{F}$, there is $\varepsilon_F > 0$ such that for any $\varepsilon \in]0; \varepsilon_F[\cap\mathbb{Q}$, $\mu_{L_\varepsilon}(\mathscr{E}'_F) < \mu_{L_\varepsilon}(\mathscr{E}')$. We set

$$\varepsilon_1 = \min\{\varepsilon_F : F \in \mathfrak{F}\} .$$

As by Lemma 2.3.19 it suffices to compare slopes for a finite set of vector subspaces, we deduce that $\varepsilon_1 > 0$. Thus, the subsheaves $\mathscr{E}'_F$ for $F \in \mathfrak{F}$ will never destabilize $\mathscr{E}'$ for $\varepsilon < \varepsilon_1$.

We then consider $F \notin \mathfrak{F}$, that is the case where $\mu_L(\mathscr{E}_F) = \mu_L(\mathscr{E})$. We then have by definition $\mathscr{E}_F \in \mathfrak{E}$. As $\mathscr{E}$ is locally free and $\mathrm{Gr}_L(\mathscr{E})$ is sufficiently smooth, we deduce that $\pi^*\mathscr{E}_F$ is saturated in $\mathscr{E}'$. Hence, $(\pi^*\mathscr{E}_F)^{\vee\vee} = \mathscr{E}'_F$ and by (4.4),

$$\mu_{L_\varepsilon}((\pi^*\mathscr{E}_F)^{\vee\vee}) - \mu_{L_\varepsilon}(\mathscr{E}'_F) = 0. \tag{4.6}$$

Therefore, for any $F \subseteq E$ such that $\mathscr{E}_F \in \mathfrak{E}$,

$$\mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}(\mathscr{E}'_F) = \mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}((\pi^*\mathscr{E}_F)^{\vee\vee}).$$

But then the sign of $\mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}(\mathscr{E}'_F)$ is given by the sign of $\mu_0(\mathscr{E}') - \mu_0(\mathscr{E}'_F)$. Again, as we only need to test for a finite number of subspaces $F \subseteq E$, we obtain the result, with $\varepsilon_0 \leq \varepsilon_1$. □

4.1.4. The case of locally trivial fibrations. We assume here that $\pi : X' \longrightarrow X$ is a locally trivial fibration. We use the notations of Section 2.1.3. Let $\mathscr{E}$ be an equivariant reflexive sheaf on X given by the family of filtrations $(E, \{E^\rho(j)\})$. As for any $\rho' \in \Sigma'(1)$, $\phi_\mathbb{R}(\rho') \in \Sigma(0) \cup \Sigma(1)$, by Corollary 4.1.3 one has $(\pi^*\mathscr{E}_F)^{\vee\vee} = \mathscr{E}'_F$ for any vector subspace F of E. According to (4.5), we get

$$\mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}(\mathscr{E}'_F) = \mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}((\pi^*\mathscr{E}_F)^{\vee\vee}). \tag{4.7}$$

Therefore, in the proof of Theorem 4.1.9, identity (4.6) holds for any vector subspace F of E. Hence, in the case of a locally trivial fibration, the assumptions on $\mathscr{E}$ and $\mathrm{Gr}_L(\mathscr{E})$ to be locally free in Theorem 4.1.9 are not necessary. Let's now consider a simple example to illustrate our results. We will assume that $X' = X$, so that the only perturbation we consider is in the polarisation from L to L'_ε.

Example 4.1.10. Let (e_1, e_2) be a basis of $\mathbb{Z}^2$. We set $u_1 = e_1$, $u_2 = e_2$, $u_3 = e_2 - 2e_1$ and $u_4 = -e_2$. Let X be the singular toric surface associated to the fan

$$\Sigma = \{0\} \cup \{\mathrm{Cone}(u_i) : 1 \leq i \leq 4\} \cup \{\mathrm{Cone}(u_i, u_{i+1}) : 1 \leq i \leq 4\} .$$

We denote by D_i the divisor corresponding to the ray $\mathrm{Cone}(u_i)$. As Σ is simplicial, the divisors D_i are $\mathbb{Q}$-Cartier. There are linear equivalences $D_1 \sim_{\mathrm{lin}} 2D_3$ and $D_2 \sim_{\mathrm{lin}} D_4 - D_3$. According to Lemma 2.1.31, we have

$$D_3 \cdot D_4 = \frac{1}{2} \qquad D_3 \cdot D_2 = \frac{1}{2} \qquad D_3 \cdot D_3 = 0 \qquad D_4 \cdot D_1 = 1 \qquad D_4 \cdot D_4 = \frac{1}{2} .$$

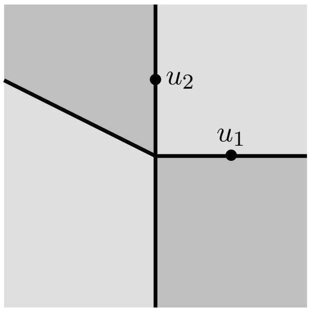


Figure 4.1: Fan of the singular surface X

Hence the $\mathbb{Q}$-divisor $aD_3 + bD_4$ is ample if and only if $a, b > 0$. As $-K_X = D_1 + D_2 + D_3 + D_4 \sim_{\text{lin}} 2(D_3 + D_4)$, we deduce that X is a del Pezzo surface. Let $\mathscr{E}$ be the tangent sheaf of X (see Example 2.3.11 for its family of filtrations). If L is an ample $\mathbb{Q}$-divisor, to check the stability of $\mathscr{E}$ with respect to L, it suffices to compare $\mu_L(\mathscr{E})$ with $\mu_L(\mathscr{E}_F)$ for $F \in \{F_1, F_2, F_3\}$ where $F_1 = \operatorname{Span}(u_1)$, $F_2 = \operatorname{Span}(u_2)$ and $F_3 = \operatorname{Span}(u_3)$. According to Example 2.3.10, we have

$$\mathscr{E}_{F_1} \cong \mathscr{O}_X(D_1) \quad , \quad \mathscr{E}_{F_2} \cong \mathscr{O}_X(D_2 + D_4) \quad \text{and} \quad \mathscr{E}_{F_3} \cong \mathscr{O}_X(D_3).$$

We assume that $L = -K_X$. We have

$$L \cdot D_1 = 2 \qquad L \cdot D_2 = 1 \qquad L \cdot D_3 = 1 \qquad L \cdot D_4 = 2$$

and

$$\mu_L(\mathscr{E}) = 3 \qquad \mu_L(\mathscr{E}_{F_1}) = 2 \qquad \mu_L(\mathscr{E}_{F_2}) = 3 \qquad \mu_L(\mathscr{E}_{F_3}) = 1\ .$$

Hence $\mathscr{E}$ is strictly semistable with respect to $-K_X$.

We now consider $L'_\varepsilon = L + \varepsilon(aD_3 + bD_4)$. From our criterion, to check stability of $\mathscr{E}$ with respect to L'_ε, it is enough to compare slopes of $\mathscr{E}$ and $\mathscr{E}_{F_2}$. We have

$$L'_\varepsilon \cdot D_1 = 2 + b\varepsilon\ , \quad L'_\varepsilon \cdot D_2 = 1 + \frac{a\varepsilon}{2}\ , \quad L'_\varepsilon \cdot D_3 = 1 + \frac{b\varepsilon}{2}\ , \quad L'_\varepsilon \cdot D_4 = 2 + \frac{(a+b)\varepsilon}{2}.$$

Thus, $\mu_{L'_\varepsilon}(\mathscr{E}) = 3 + \left(b + \frac{a}{2}\right)\varepsilon$, and $\mu_{L'_\varepsilon}(\mathscr{E}_{F_2}) = 3 + \left(a + \frac{b}{2}\right)\varepsilon$. We deduce that $\mathscr{E}$ is stable (resp. strictly semistable) with respect to L'_ε if and only if $b - a > 0$ (resp. $b - a = 0$). ♢

4.2. Blow-ups

In this section we specialize to equivariant blow-ups along smooth centers. Let X be a smooth toric variety of dimension n associated to a smooth fan Σ. We denote by $\pi : X' \longrightarrow X$ the blowup of X along $Z = V(\tau)$ with $\tau \in \Sigma$ such that $\dim \tau \geq 2$ and we set $\Sigma' = \Sigma^*(\tau)$. As before, $\mathscr{E}$ stands for an equivariant reflexive sheaf on X and $\mathscr{E}'$ denotes its reflexive pullback along π.

4.2.1. Slope of the reflexive pullback along a blowup. In this section, we show Proposition 4.2.1 which is the key to the results stated in this part. Note that the proof doesn't require any toric assumption.

Proposition 4.2.1. *Let X be a smooth projective variety and $Z \subsetneq X$ a smooth irreducible subvariety of dimension ℓ with $1 \leq \ell \leq \dim(X) - 2$. We denote by $\pi : X' \longrightarrow X$ the blowup of X along*

Z and D_0 the exceptional divisor of π. Let L be an ample divisor of X and let $L_\varepsilon = \pi^*L - \varepsilon D_0$ be an ample $\mathbb{Q}$-divisor of X' for $\varepsilon \in \mathbb{Q}_{>0}$ small. Then for any divisor D of X,

$$\pi^*D \cdot L_\varepsilon^{n-1} = D \cdot L^{n-1} - \binom{n-1}{\ell-1}\varepsilon^{n-\ell} D \cdot L^{\ell-1} \cdot Z + o(\varepsilon^{n-\ell}) \tag{4.8}$$

and

$$D_0 \cdot L_\varepsilon^{n-1} = \binom{n-1}{\ell}\varepsilon^{n-\ell-1} Z \cdot L^{\ell} + o(\varepsilon^{n-\ell-1}). \tag{4.9}$$

Proof. We denote by $\mathscr{N}$ the normal bundle of Z. We have $D_0 = \mathbb{P}(\mathscr{N})$. For a divisor D of X, one has

$$\begin{aligned}\pi^*D \cdot L_\varepsilon^{n-1} &= \sum_{k=0}^{n-1}\binom{n-1}{k}\pi^*D \cdot (\pi^*L)^k \cdot (-\varepsilon D_0)^{n-1-k}\\ &= \pi^*D \cdot (\pi^*L)^{n-1} + \sum_{k=0}^{n-2}\binom{n-1}{k}\pi^*D \cdot (\pi^*L)^k \cdot (-\varepsilon D_0)^{n-1-k}.\end{aligned}$$

Therefore, by the projection formula, we get

$$\pi^*D \cdot L_\varepsilon^{n-1} = D \cdot L^{n-1} + \sum_{k=0}^{n-2}\binom{n-1}{k}(-\varepsilon)^{n-1-k} D \cdot L^k \cdot \pi_*(D_0^{n-1-k}).$$

If $\eta = \pi_{|D_0}$, according to [7, Example 3.3.4], one has

$$\sum_{k\geq 1}(-1)^{k-1}\eta_*(D_0^k) = s(\mathscr{N}) \cap [Z]$$

where $s(\mathscr{N})$ is the total Segre class of $\mathscr{N}$. As $s_i(\mathscr{N}) \cap [Z] \in A_{\ell-i}(Z)$ and $\eta_*(D_0^{n-k-1}) \in A_{k+1}(Z)$, we deduce that

$$(-1)^{n-k}\eta_*(D_0^{n-k-1}) = s_{\ell-1-k}(\mathscr{N}) \cap [Z]$$

for any $k \in \{0, \dots, n-2\}$. As $s_0(\mathscr{N}) \cap [Z] = [Z]$, we get

$$\begin{aligned}\pi^*D \cdot L_\varepsilon^{n-1} = D \cdot L^{n-1} &- \binom{n-1}{\ell-1}\varepsilon^{n-\ell} D \cdot L^{\ell-1} \cdot Z\\ &- \sum_{k=0}^{l-2}\binom{n-1}{k}\varepsilon^{n-1-k} D \cdot L^k \cdot (s_{\ell-1-k}(\mathscr{N}) \cap [Z])\end{aligned}$$

which gives the first formula. □

Corollary 4.2.2. *With the same data as Proposition 4.2.1, if $\mathscr{E}$ is a reflexive sheaf on X, then*

$$\mu_{L_\varepsilon}((\pi^*\mathscr{E})^{\vee\vee}) = \mu_L(\mathscr{E}) - \binom{n-1}{\ell-1}\mu_{L_{|Z}}(\mathscr{E}_{|Z})\varepsilon^{n-\ell} + O(\varepsilon^{n-\ell+1}).$$

4.2.2. Reflexive pullback along an equivariant blow-up. In this section we give the family of filtrations of the reflexive pullback along an equivariant blowup. This will serve in relating the Chern classes, and also in obtaining explicit examples. Let $(u_1, \ldots, u_n)$ be a basis of N such that $\tau = \mathrm{Cone}(u_1, \ldots, u_s)$ with $2 \leq s \leq n$ and $\{\mathrm{Cone}(A) : A \subseteq \{u_1, \ldots, u_n\}\} \subseteq \Sigma$. We set $\rho_i = \mathrm{Cone}(u_i)$ for $i \in \{1, \ldots, s\}$ and $\rho_0 = \mathrm{Cone}(u_\tau)$ where $u_\tau = u_1 + \ldots + u_s$. We denote by $(e_1, \ldots, e_n)$ the dual basis of $(u_1, \ldots, u_n)$.

Remark 4.2.3. The variety $V(\tau)$ is the center of the blowup $\pi : X' \longrightarrow X$ and D_{ρ_0} is the exceptional divisor of π.

Proposition 4.2.4. *Let $\mathscr{E}$ be an equivariant reflexive sheaf on X given by the family of filtrations $(E, \{E^\rho(j)\})$. Let $\left(E, \{\widetilde{E}^\rho(j)\}_{\rho\in\Sigma'(1), j\in\mathbb{Z}}\right)$ be the family of filtrations of $\mathscr{E}' = (\pi^*\mathscr{E})^{\vee\vee}$. Then we have :*

1. *if $\rho \in \Sigma(1) \subseteq \Sigma'(1)$, $\widetilde{E}^\rho(j) = E^\rho(j)$;*
2. *if $\rho = \rho_0$,*
$$\widetilde{E}^\rho(j) = \sum_{i_1+\ldots+i_s=j} E^{\rho_1}(i_1) \cap \ldots \cap E^{\rho_s}(i_s) \,.$$

Proof. We recall that the $\mathbb{Z}$-linear map $\phi = \mathrm{Id}_N$ is compatible with Σ' and Σ. If $\rho \in \Sigma(1) \subseteq \Sigma'(1)$, we have $\phi(u_\rho) = u_\rho$. By Proposition 4.1.1 we get $\widetilde{E}^\rho(j) = E^\rho(j)$.

We now assume that $\rho = \mathrm{Cone}(u_\tau)$. The minimal cone of Σ which contains $\phi_{\mathbb{R}}(\rho)$ is τ. Hence by Lemma 2.1.21, we deduce that $\pi\left(O(\rho)\right) = O(\tau)$. Thus, $\pi\left(U'_\rho\right) = T \cup O(\tau)$. As U_τ is the minimal T-invariant open subset of X which contains $\pi\left(U'_\rho\right)$, we deduce that $\Gamma(U'_\rho, \pi^{-1}\mathscr{E}) = \Gamma(U_\tau, \mathscr{E})$. By (4.1) we get

$$\widetilde{E}^\rho = \Gamma(U'_\rho, \mathscr{E}') = \Gamma(U'_\rho, \pi^{-1}\mathscr{E}) \otimes_{\mathscr{O}_X(U_\tau)} \mathscr{O}_{X'}(U'_\rho) = E^\tau \otimes_{\mathscr{O}_X(U_\tau)} \mathscr{O}_{X'}(U'_\rho)$$

where $\mathscr{O}_{X'}(U'_\rho) = \mathbb{C}[S_\rho]$, $\mathscr{O}_X(U_\tau) = \mathbb{C}[S_\tau]$ and E^τ defined in Notation 2.3.9. We have

$$\tau^\vee = \mathrm{Cone}(e_1, \ldots, e_s, \pm e_{s+1}, \ldots, \pm e_n) \,.$$

A point $m = m_1\, e_1 + \ldots + m_n\, e_n$ lies in $\rho^\vee$ if and only if $m_1 + \ldots + m_s \geq 0$, i.e $m_s \geq -(m_1 + \ldots + m_{s-1})$. Hence,

$$\rho^\vee = \mathrm{Cone}(\pm(e_1 - e_s), \ldots, \pm(e_{s-1} - e_s), e_s, \pm e_{s+1}, \ldots, \pm e_n)$$

and

$$\rho^\perp = \mathrm{Cone}(\pm(e_1 - e_s), \ldots, \pm(e_{n-1} - e_s), \pm e_{s+1}, \ldots, \pm e_n) \,.$$

Therefore, $S_\rho = \rho^\perp + S_\tau$ and $\mathbb{C}[S_\rho] = \mathbb{C}[S_\tau] \otimes_{\mathbb{C}} \mathbb{C}[M(\rho)]$. Thus,

$$\widetilde{E}^\rho = E^\tau \otimes_{\mathbb{C}[S_\tau]} (\mathbb{C}[S_\tau] \otimes_{\mathbb{C}} \mathbb{C}[M(\rho)]) = E^\tau \otimes_{\mathbb{C}} \mathbb{C}[M(\rho)] \,.$$

Hence,

$$\begin{aligned}
\widetilde{E}^\rho = \sum_{m'\in M(\rho)} E^\tau \otimes \chi^{m'} &= \sum_{m'\in M(\rho)} \left(\sum_{m\in M} E^\tau_m \otimes \chi^m\right) \otimes \chi^{m'} \\
&= \sum_{m'\in M(\rho)} \left(\sum_{m\in M} E^\tau_{m-m'} \otimes \chi^m\right) \\
&= \sum_{m\in M} \left(\sum_{m'\in M(\rho)} E^\tau_{m-m'}\right) \otimes \chi^m
\end{aligned}$$

Therefore, for any $m \in M$,

$$\widetilde{E}^{\rho}_{m} = \sum_{m' \in M(\rho)} E^{\tau}_{m-m'} .$$

As for any $m' \in M(\rho)$, $\langle m - m', u_1 \rangle + \ldots + \langle m - m', u_s \rangle = \langle m - m', u_\tau \rangle = \langle m, u_\tau \rangle$, by using the fact that $E^{\tau}_{m-m'} = E^{\rho_1}(\langle m - m', u_1 \rangle) \cap \ldots \cap E^{\rho_s}(\langle m - m', u_s \rangle)$ and $\widetilde{E}^{\rho}_{m} = \widetilde{E}^{\rho}(\langle m, u_\tau \rangle)$, we get the result. □

The following example shows that the reflexive pullback of $\mathscr{E}_F$ might not be saturated in $\mathscr{E}'$ in general. Hence, our hypotheses on $\mathscr{E}$ being sufficiently smooth, or on pulled back subsheaves being saturated, are necessary in the statements of our results.

Example 4.2.5. Let (u_1, u_2) be a basis of $\mathbb{Z}^2$ and $\Sigma = \{\mathrm{Cone}(A) : A \subseteq \{u_1, u_2\}\}$. Let $\mathscr{E}$ be an equivariant reflexive sheaf on $X = \mathbb{C}^2$ given by the family of filtrations

$$E^{\rho_1}(j) = \begin{cases} \{0\} & \text{if } j \leq 0 \\ E_1 & \text{if } 1 \leq j \leq 2 \\ E & \text{if } j \geq 3 \end{cases} \quad \text{and} \quad E^{\rho_2}(j) = \begin{cases} \{0\} & \text{if } j \leq 0 \\ E_2 & \text{if } j = 1 \\ E & \text{if } j \geq 2 \end{cases}$$

where $E_1 = \mathrm{Span}(u_1)$, $E_2 = \mathrm{Span}(u_2)$ and $E = \mathrm{Span}(u_1, u_2)$. We denote by $\pi : X' \longrightarrow X$ the blowup along $V(\mathrm{Cone}(u_1, u_2))$ (that is the blowup at the origin). We set $F = \mathrm{Span}(u_1 + u_2)$ and $\mathscr{E}_F$ the subsheaf of $\mathscr{E}$ given by $F^{\rho}(j) = E^{\rho}(j) \cap F$. According to Proposition 4.2.4,

$$\widetilde{E}^{\rho_0}(j) = \begin{cases} \{0\} & \text{if } j \leq 2 \\ E_1 & \text{if } j = 3 \\ E & \text{if } j \geq 4 \end{cases} \quad \text{and} \quad \widetilde{F}^{\rho_0}(j) = \begin{cases} \{0\} & \text{if } j \leq 4 \\ F & \text{if } j \geq 5 \end{cases} .$$

As $\widetilde{F}^{\rho_0}(4) \neq \widetilde{E}^{\rho_0}(4) \cap F$, we deduce that $(\pi^* \mathscr{E}_F)^{\vee\vee}$ is not saturated in $(\pi^* \mathscr{E})^{\vee\vee}$. ◊

Let $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ be a Cartier divisor of X. According to Proposition 2.1.24, we have

$$\pi^* D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho + \sum_{\rho \in \tau(1)} a_\rho D_0 . \tag{4.10}$$

As $c_1(\mathscr{E}') = \pi^* c_1(\mathscr{E})$, we get

$$c_1(\mathscr{E}') = - \sum_{\rho \in \Sigma(1)} \iota_\rho(\mathscr{E}) D_\rho - \sum_{\rho \in \tau(1)} \iota_\rho(\mathscr{E}) D_0 . \tag{4.11}$$

In the following Lemma we give the expression of $c_1(\mathscr{E}'_F)$ with respect to $c_1(\mathscr{E}_F)$.

Lemma 4.2.6. *Let F be a vector subspace of E. The first Chern class of $\mathscr{E}'_F$ is given by*

$$c_1(\mathscr{E}'_F) = \pi^* c_1(\mathscr{E}_F) + \sum_{j \in \mathbb{Z}} d_j(F)\, D_0$$

where $d_j(F) = \dim(F \cap \widetilde{E}^{\rho_0}(j)) - \dim \widetilde{F}^{\rho_0}(j)$.

Proof. By Corollary 4.1.3, if $\rho \in \Sigma(1)$, we have $F \cap \widetilde{E}^{\rho}(j) = F^{\rho}(j)$. Thus, for any $\rho \in \Sigma(1)$, $\iota_\rho(\mathscr{E}'_F) = \iota_\rho(\mathscr{E}_F)$. We now consider the case $\rho = \rho_0$. We have

$$\iota_{\rho_0}(\mathscr{E}'_F) = \sum_{j \in \mathbb{Z}} j \left(\dim(F \cap \widetilde{E}^{\rho_0}(j)) - \dim(F \cap \widetilde{E}^{\rho_0}(j-1)) \right)$$

and

$$\begin{aligned}\iota_{\rho_0}(\mathscr{E}'_F) - \iota_{\rho_0}(\pi^*\mathscr{E}_F) &= \sum_{j\in\mathbb{Z}} j\left(\dim(F\cap \widetilde{E}^{\rho_0}(j)) - \dim \widetilde{F}^{\rho_0}(j)\right) \\ &\quad - \sum_{j\in\mathbb{Z}} j\left(\dim(F\cap \widetilde{E}^{\rho_0}(j-1)) - \dim \widetilde{F}^{\rho_0}(j-1)\right) \\ &= \sum_{j\in\mathbb{Z}} j\, d_j(F) - \sum_{j\in\mathbb{Z}} j\, d_{j-1}(F)\end{aligned}$$

There are $p,\, q\in\mathbb{Z}$ with $p<q$ such that $d_j(F)=0$ if $j<p$ and $d_j(F)=0$ if $j>q$. Hence,

$$\iota_{\rho_0}(\mathscr{E}'_F) - \iota_{\rho_0}(\pi^*\mathscr{E}_F) = \sum_{j=p}^{q} j\, d_j(F) - \sum_{j=p+1}^{q+1} j\, d_{j-1}(F) = -\sum_{j\in\mathbb{Z}} d_j(F)$$

and $c_1(\mathscr{E}'_F) = c_1(\pi^*\mathscr{E}_F) + \left(\sum_{j\in\mathbb{Z}} d_j(F)\right) D_0$. □

Corollary 4.2.7. *Let F be a vector subspace of E. The sheaf $(\pi^*\mathscr{E}_F)^{\vee\vee}$ is saturated in $\mathscr{E}'$ if and only if $(\pi^*\mathscr{E}_F)^{\vee\vee} = \mathscr{E}'_F$; in that case, $c_1(\mathscr{E}'_F) = \pi^* c_1(\mathscr{E}_F)$ and $d_j(F) = 0$ for all $j\in\mathbb{Z}$.*

Proof. The proof follows from Lemma 2.3.17. □

For a semistable sheaf $\mathscr{E}$, we recall that $\mathfrak{E}$ is the set of equivariant saturated subsheaves of $\mathscr{E}$ arising in a Jordan-Holder filtration for $\mathscr{E}$ (cf. Section 4.1.3). By Corollary 4.2.2 and Lemma 4.2.6, we get:

Corollary 4.2.8. *Let (X, L) be a smooth polarized toric variety. Let $\pi : X' \longrightarrow X$ be the blowup along a T-invariant irreducible subvariety $Z\subseteq X$ with $1\le \dim(Z)\le \dim(X)-2$ and let $L_\varepsilon = \pi^*L - \varepsilon E$ for E the exceptional divisor of π and $\varepsilon\in\mathbb{Q}_{>0}$ small. Let $\mathscr{E}$ be a T-equivariant reflexive sheaf that is strictly semistable on (X, L). Assume that for all $\mathscr{F}\in\mathfrak{E}$, $(\pi^*\mathscr{F})^{\vee\vee}$ is saturated in $\mathscr{E}' := (\pi^*\mathscr{E})^{\vee\vee}$ and that*

$$\mu_{L_{|Z}}(\mathscr{E}_{|Z}) < \mu_{L_{|Z}}(\mathscr{F}_{|Z}).$$

Then there is $\varepsilon_0>0$ such that for all $\varepsilon\in]0,\varepsilon_0[\cap\mathbb{Q}$, the pullback $\mathscr{E}'$ is stable on (X', L_ε).

4.2.3. Blowup in several points.

In this section, we prove the following theorem where $\mathfrak{E}$ is the set of equivariant saturated subsheaves of $\mathscr{E}$ arising in a Jordan-Holder filtration of $\mathscr{E}$.

Theorem 4.2.9. *Let (X, L) be a smooth polarized toric variety and S a set of invariant points under the torus action of X. Let $\pi : X' \longrightarrow X$ be the blowup along S and let $L_\varepsilon = \pi^*L - \varepsilon E$ for E the exceptional divisor of π and $\varepsilon\in\mathbb{Q}_{>0}$ small. Let $\mathscr{E}$ be a T-equivariant reflexive sheaf that is strictly semistable on (X, L). Then there is $\varepsilon_0>0$ such that for all $\varepsilon\in]0,\varepsilon_0[\cap\mathbb{Q}$, the reflexive pullback $\mathscr{E}' := (\pi^*\mathscr{E})^{\vee\vee}$ on (X', L_ε) is*

1. *strictly semistable iff for any subsheaf $\mathscr{F}\in\mathfrak{E}$, $(\pi^*\mathscr{F})^{\vee\vee}$ is saturated in $\mathscr{E}'$,*
2. *unstable otherwise.*

Proof. For any $p\in S$ there is $\sigma\in\Sigma(n)$ such that $p=\gamma_\sigma$. We set $S_\Sigma = \{\sigma\in\Sigma(n) : \gamma_\sigma\in S\}$. According to Section 2.1.3, the fan Σ' of X' is given by

$$\Sigma' = \{\sigma\in\Sigma : \sigma\notin S_\Sigma\} \cup \bigcup_{\sigma\in S_\Sigma} \Sigma^*_\sigma(\sigma).$$

We denote by $D_p \subseteq X'$ the exceptional divisor of π over $p \in S$; we have $E = \sum_{p \in S} D_p$.

Let $\sigma = \mathrm{Cone}(u_1, \ldots, u_n) \in S_\Sigma$ and $p = \gamma_\sigma$. We denote by $(e_1, \ldots, e_n)$ the dual basis of $(u_1, \ldots, u_n)$ and we set $\rho_i = \mathrm{Cone}(u_i)$. We compute the intersection product on X'. We have $[D_\rho] \cdot [D_p] = 0$ if $\rho \in \Sigma'(1) \setminus (\sigma(1) \cup \{\mathrm{Cone}(u_1 + \ldots + u_n)\})$. For $i \in \{1, \ldots, n\}$, if we set $m = -e_i$, by Lemma 2.1.31 we get

$$[D_p] \cdot [D_p] = [D_p + \mathrm{div}(\chi^m)] \cdot [D_p] = -[D_{\rho_i}] \cdot [D_p];$$

therefore

$$\begin{cases} D_p^n = (-1)^{n-1} & \\ D_\rho \cdot D_p^{n-1} = (-1)^n & \text{if } \rho \in \sigma(1) \\ D_\rho \cdot D_p^{n-1} = 0 & \text{if } \rho \in \Sigma'(1) \setminus (\sigma(1) \cup \{\mathrm{Cone}(u_\sigma)\}) \end{cases}.$$

If $L = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$, by (4.10), we get

$$\pi^* L = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho + \sum_{\sigma \in S_\Sigma} \sum_{\rho \in \sigma(1)} a_\rho D_{\gamma_\sigma} \ ;$$

hence, for any $p \in S$, $[\pi^* L] \cdot [D_p] = 0 \in A_{n-2}(X')$. Thus,

$$L_\varepsilon^{n-1} = (\pi^* L)^{n-1} + (-1)^{n-1} \varepsilon^{n-1} \sum_{p \in S} D_p^{n-1}.$$

For any $p \in S$, we have $\deg_{L_\varepsilon}(D_p) = \varepsilon^{n-1}$. If $\rho \in \Sigma(1)$, then

$$\deg_{L_\varepsilon}(D_\rho) = \deg_L(D_\rho) - \sum_{\sigma \in S_\Sigma,\, \rho \in \sigma(1)} \varepsilon^{n-1}.$$

Thus,

$$\begin{aligned} \mathrm{rk}(\mathscr{E}')\mu_{L_\varepsilon}(\mathscr{E}') &= -\sum_{\rho \in \Sigma(1)} \iota_\rho(\mathscr{E}) \deg_{L_\varepsilon}(D_\rho) - \sum_{\sigma \in S_\Sigma} \sum_{\rho \in \sigma(1)} \iota_\rho(\mathscr{E}) \deg_{L_\varepsilon}(D_{\gamma_\sigma}) \\ &= -\sum_{\rho \in \Sigma(1)} \iota_\rho(\mathscr{E}) \deg_L(D_\rho) - \sum_{\sigma \in S_\Sigma} \sum_{\rho \in \sigma(1)} \iota_\rho(\mathscr{E}) \varepsilon^{n-1} \\ &\quad + \sum_{\rho \in \Sigma(1)} \iota_\rho(\mathscr{E}) \left(\sum_{\sigma \in S_\Sigma,\, \rho \in \sigma(1)} \varepsilon^{n-1} \right) \\ &= \mathrm{rk}(\mathscr{E})\mu_L(\mathscr{E}). \end{aligned}$$

Hence, $\mu_{L_\varepsilon}(\mathscr{E}') = \mu_L(\mathscr{E})$. If F is a vector subspace of E, the same computation gives

$$\mu_{L_\varepsilon}((\pi^* \mathscr{E}_F)^{\vee\vee}) = \mu_L(\mathscr{E}_F).$$

According to Lemma 4.2.6, for any vector subspace F of E one has

$$\mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}(\mathscr{E}'_F) = \mu_L(\mathscr{E}) - \mu_L(\mathscr{E}_F) - \frac{\varepsilon^{n-1}}{\mathrm{rk}(\mathscr{E}_F)} \sum_{p \in S} \sum_{j \in \mathbb{Z}} (\dim(F \cap \widetilde{E}^{\rho_p}(j)) - \dim \widetilde{F}^{\rho_p}(j))$$

where ρ_p is the ray corresponding to the divisor D_p. Hence, Theorem 4.2.9 follows from Corollary 4.2.7. □

4.2.4. Blowup along a curve. In this section, we assume that $n = \dim(X) \geq 3$ and that $\tau \in \Sigma(n-1)$ is the intersection of two n-dimensional cones σ and σ'. Hence we consider the blowup $\pi : X' \longrightarrow X$ along the curve $Z = V(\tau)$. With the results of Section 4.2.1, we can prove the following theorem.

Theorem 4.2.10. *Let (X, L) be a smooth polarised toric variety. Let $\pi : X' \longrightarrow X$ be the blow-up along a T-invariant irreducible curve $Z \subseteq X$ and let $L_\varepsilon = \pi^* L - \varepsilon E$ for E the exceptional divisor of π and $\varepsilon \in \mathbb{Q}_{>0}$ small. Let $\mathscr{E}$ be a T-equivariant reflexive sheaf that is strictly semistable on (X, L). Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \in]0, \varepsilon_0[\cap\mathbb{Q}$, the pullback $\mathscr{E}' := (\pi^*\mathscr{E})^{\vee\vee}$ on (X', L_ε) is*

1. *stable iff for all $\mathscr{F} \in \mathfrak{E}$, $(\pi^*\mathscr{F})^{\vee\vee}$ is saturated in $\mathscr{E}'$ and*
$$\frac{c_1(\mathscr{E}) \cdot Z}{\operatorname{rk} \mathscr{E}} < \frac{c_1(\mathscr{F}) \cdot Z}{\operatorname{rk} \mathscr{F}};$$
2. *semistable iff for all $\mathscr{F} \in \mathfrak{E}$, $(\pi^*\mathscr{F})^{\vee\vee}$ is saturated in $\mathscr{E}'$ and*
$$\frac{c_1(\mathscr{E}) \cdot Z}{\operatorname{rk} \mathscr{E}} \leq \frac{c_1(\mathscr{F}) \cdot Z}{\operatorname{rk} \mathscr{F}};$$
3. *unstable otherwise.*

Proof. Let $\mathscr{E}$ be a strictly semistable sheaf on $(X,\ L)$. According to Corollary 4.2.2 one has
$$\mu_{L_\varepsilon}(\mathscr{E}') = \mu_L(\mathscr{E}) - \frac{\varepsilon^{n-1}}{\operatorname{rk}(\mathscr{E})} c_1(\mathscr{E}) \cdot V(\tau)$$
and by (4.9) we have
$$\deg_{L_\varepsilon}(D_0) = (n-1)\varepsilon^{n-2} L \cdot V(\tau) - (-1)^n \varepsilon^{n-1} D_0^n \,.$$
By Lemma 4.2.6, for any vector subspace F of E, we have
$$\mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}(\mathscr{E}'_F) = \mu_L(\mathscr{E}) - \mu_L(\mathscr{E}_F) + \varepsilon^{n-1} \left(\frac{c_1(\mathscr{E}_F) \cdot V(\tau)}{\operatorname{rk}(\mathscr{E}_F)} - \frac{c_1(\mathscr{E}) \cdot V(\tau)}{\operatorname{rk}(\mathscr{E})} \right)$$
$$- \frac{\varepsilon^{n-2}}{\operatorname{rk}(\mathscr{E}_F)} \left((n-1) L \cdot V(\tau) - (-1)^n \varepsilon\, D_0^n \right) \sum_{j \in \mathbb{Z}} d_j(F) \,. \quad (4.12)$$
Let $\mathscr{E}_F \in \mathfrak{E}$ for $F \subsetneq E$. We set $C = \sum_j d_j(F)$. If $(\pi^*\mathscr{E}_F)^{\vee\vee}$ is not saturated in $\mathscr{E}'$, by Corollary 4.2.7, we have $C > 0$; hence there is $\varepsilon_F > 0$ such that for any $\varepsilon \in]0, \varepsilon_F[\cap\mathbb{Q}$,
$$\varepsilon \left(\frac{c_1(\mathscr{E}_F) \cdot V(\tau)}{\operatorname{rk}(\mathscr{E}_F)} - \frac{c_1(\mathscr{E}) \cdot V(\tau)}{\operatorname{rk}(\mathscr{E})} + \frac{(-1)^n C \times D_0^n}{\operatorname{rk}(\mathscr{E}_F)} \right) < \frac{(n-1) C \times L \cdot V(\tau)}{\operatorname{rk}(\mathscr{E}_F)} \,.$$
If $(\pi^*\mathscr{E}_F)^{\vee\vee}$ is saturated in $\mathscr{E}'$, then for $0 < \varepsilon \ll 1$, $\mu_{L_\varepsilon}(\mathscr{E}') - \mu_{L_\varepsilon}(\mathscr{E}'_F) > 0$ (resp. ≥ 0) if and only if
$$\frac{c_1(\mathscr{E}_F) \cdot V(\tau)}{\operatorname{rk}(\mathscr{E}_F)} - \frac{c_1(\mathscr{E}) \cdot V(\tau)}{\operatorname{rk}(\mathscr{E})} > 0 \text{ (resp. } \geq 0) \,.$$
With these two observations about $(\pi^*\mathscr{E}_F)^{\vee\vee}$ for $\mathscr{E}_F \in \mathfrak{E}$, we deduce the result. ☐

We turn now to an explicit formula that helps applying Theorem 4.2.10 on concrete examples. For a divisor D of X, we can compute $D \cdot V(\tau)$ by using the fact that $\tau = \sigma \cap \sigma'$. Let $\Sigma_0 = \sigma(1) \cup \sigma'(1)$. There is a family of numbers $\alpha_\rho \in \mathbb{Z}$ such that
$$\sum_{\rho \in \Sigma_0} \alpha_\rho u_\rho = 0 \quad \text{and} \quad \alpha_\rho = 1 \text{ if } \rho \in \Sigma_0 \setminus \tau(1) \,.$$

We assume that $\sigma = \mathrm{Cone}(u_1, \dots, u_n)$, $\sigma' = \mathrm{Cone}(u_1, \dots, u_{n-1}, u_{n+1})$ and $\Sigma_0 = \{\mathrm{Cone}(u_i) : 1 \leq i \leq n+1\}$. For $i \in \{1, \dots, n+1\}$, we set $\rho_i = \mathrm{Cone}(u_i)$ and $\alpha_i = \alpha_{\rho_i}$. We denote by $(e_1, \dots, e_n)$ the dual basis of $(u_1, \dots, u_n)$. For $i \in \{1, \dots, n-1\}$, we have

$$D_{\rho_i} \sim_{\mathrm{lin}} D_{\rho_i} + \mathrm{div}(\chi^{-e_i}) = \alpha_i\, D_{\rho_{n+1}} + \sum_{\rho \in \Sigma(1) \setminus \Sigma_0} \langle -e_i, u_\rho \rangle D_\rho \,.$$

By Proposition 2.1.29, we get

$$D_\rho \cdot V(\tau) = \begin{cases} \alpha_\rho & \text{if } \rho \in \Sigma_0 \\ 0 & \text{if } \rho \in \Sigma(1) \setminus \Sigma_0 \end{cases} .$$

Hence,

$$\frac{c_1(\mathscr{E}_F) \cdot V(\tau)}{\mathrm{rk}(\mathscr{E}_F)} - \frac{c_1(\mathscr{E}) \cdot V(\tau)}{\mathrm{rk}(\mathscr{E})} = \sum_{\rho \in \Sigma_0} \alpha_\rho \left(\frac{\iota_\rho(\mathscr{E})}{\mathrm{rk}(\mathscr{E})} - \frac{\iota_\rho(\mathscr{E}_F)}{\mathrm{rk}(\mathscr{E}_F)} \right) . \tag{4.13}$$

4.2.5. Examples of (de)stabilizing blow-ups along curves. We use the notation of Section 2.2.2. Let $X = X_\Sigma$ be a smooth toric variety of dimension n given by

$$X = \mathbb{P}\left(\mathscr{O}_{\mathbb{P}^1}^{\oplus r} \oplus \mathscr{O}_{\mathbb{P}^1}(1)\right)$$

with $r \geq 2$ such that $r+1 = n$. We denote by $\mathrm{pr} : X \longrightarrow \mathbb{P}^1$ the projection map. Let $\mathscr{E}$ be the tangent sheaf of X. The family of filtrations of $\mathscr{E}$ is given in Example 2.3.11. According to [14, Theorem 1.4], the sheaf $\mathscr{E}$ is stable with respect to $L = \mathrm{pr}^* \mathscr{O}_{\mathbb{P}^1}(\nu) \otimes \mathscr{O}_X(1)$ if and only if $0 < \nu < \nu_0$ with $\nu_0 = \frac{1}{r+1}$.

We now assume that $L = \mathrm{pr}^* \mathscr{O}_{\mathbb{P}^1}(1/(r+1)) \otimes \mathscr{O}_X(1)$. The sheaf $\mathscr{E}$ is strictly semistable with respect to L. The subsheaf $\mathscr{E}_F$ with $F = \mathrm{Span}(v_0, \dots, v_r)$ is the unique saturated subsheaf of $\mathscr{E}$ such that $\mu_L(\mathscr{E}_F) = \mu_L(\mathscr{E})$. The family of filtrations of $\mathscr{E}_F$ is given by

$$F^\rho(j) = \begin{cases} 0 & \text{if } j < -1 \\ \mathrm{Span}(u_\rho) & \text{if } j = -1 \\ F & \text{if } j > -1 \end{cases} \quad \text{if } \rho = \mathrm{Cone}(v_i)$$

and by

$$F^\rho(j) = \begin{cases} 0 & \text{if } j < 0 \\ F & \text{if } j \geq 0 \end{cases} \quad \text{if } \rho = \mathrm{Cone}(w_j) \,.$$

Hence,

$$\frac{\iota_\rho(\mathscr{E})}{r+1} - \frac{\iota_\rho(\mathscr{E}_F)}{r} = \begin{cases} \frac{1}{r} - \frac{1}{r+1} & \text{if } \rho = \mathrm{Cone}(v_i) \\ \frac{-1}{r+1} & \text{if } \rho = \mathrm{Cone}(w_j) \end{cases} .$$

Given $\tau \in \Sigma(n-1)$, in the following examples, we study the stability of the reflexive pull-back $\mathscr{E}' = (\pi^*\mathscr{E})^{\vee\vee}$ on $X' = \mathrm{Bl}_{V(\tau)}(X)$ with respect to small perturbations of $\pi^* L$. In these examples, $(\pi^*\mathscr{E}_F)^{\vee\vee}$ is saturated in $\mathscr{E}'$.

Example 4.2.11. Let $\tau = \mathrm{Cone}(w_0, v_1, \dots, v_{r-1})$. We have

$$\tau = \mathrm{Cone}(w_0, v_1, \dots, v_{r-1}, v_r) \cap \mathrm{Cone}(w_0, v_1, \dots, v_{r-1}, v_0) \,.$$

As $0 \cdot w_0 + v_0 + v_1 + \ldots + v_r = 0$, by Equation (4.13) we get

$$\frac{c_1(\mathscr{E}_F) \cdot V(\tau)}{r} - \frac{c_1(\mathscr{E}) \cdot V(\tau)}{r+1} = \frac{r+1}{r} - 1$$

So there is $\varepsilon_0 > 0$ such that for any $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, $\mathscr{E}'$ is stable with respect to L_ε. $\diamondsuit$

Example 4.2.12. Let $\tau = \mathrm{Cone}(v_0,\ v_1, \ldots, v_{r-1})$. We have

$$\tau = \mathrm{Cone}(v_0,\ v_1, \ldots, v_{r-1},\ w_0) \cap \mathrm{Cone}(v_0,\ v_1, \ldots, v_{r-1},\ w_1)$$

As $w_0 + w_1 + v_0 + v_1 + \ldots + v_{r-1} = 0$, by Equation (4.13) we get

$$\frac{c_1(\mathscr{E}_F) \cdot V(\tau)}{r} - \frac{c_1(\mathscr{E}) \cdot V(\tau)}{r+1} = \frac{-1}{r+1}\ .$$

Hence, there is $\varepsilon_0 > 0$ such that for any $\varepsilon \in]0, \varepsilon_0[\cap \mathbb{Q}$, $\mathscr{E}'$ is unstable with respect to L_ε. $\Diamond$

5

ON THE SINGULAR LOCUS OF TORIC SHEAVES

We study the singular locus of an equivariant reflexive sheaf over a smooth toric variety. This locus is a finite union of orbit closures of codimension at least three. In the other direction, we show that it is possible to prescribe singularities on a sheaf.

5.1. Prescribing singularities

5.1.1. Dimension of the singular locus. Let X be a smooth toric variety with torus T associated to a fan Σ and $\mathscr{E}$ an equivariant reflexive sheaf on X given by the family of filtrations $(E, \{E^\rho(j)\})$. We recall that N is the lattice of one-parameter subgroups of T and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$.

Notation 5.1.1. We denote by:
1. $\Sigma(\mathscr{E})_{\mathrm{free}}$ the set of cones $\sigma \in \Sigma$ such that $\mathscr{E}$ is locally free on U_σ,
2. $\Sigma(\mathscr{E})_{\mathrm{sing}} = \Sigma \setminus \Sigma(\mathscr{E})_{\mathrm{free}}$,
3. $X(\mathscr{E})_{\mathrm{free}}$ the smooth locus of $\mathscr{E}$,
4. $X(\mathscr{E})_{\mathrm{sing}} = X \setminus X(\mathscr{E})_{\mathrm{free}}$ the singular locus of $\mathscr{E}$.

From general theory, the singular locus $X(\mathscr{E})_{\mathrm{sing}}$ is a Zariski closed subset of X of codimension at least 3 [24, Corollary 5.5.20]. It is not difficult to see that $\Sigma(\mathscr{E})_{\mathrm{free}}$ and $\Sigma(\mathscr{E})_{\mathrm{sing}}$ satisfy the following result.

Lemma 5.1.2. *Let $\mathscr{E}$ be an equivariant reflexive sheaf on a smooth toric variety X. Then:*
1. *The set $\Sigma(\mathscr{E})_{\mathrm{free}}$ is a subfan of Σ.*
2. *If $\tau \in \Sigma(\mathscr{E})_{\mathrm{sing}}$ is a face of $\sigma \in \Sigma$, then $\sigma \in \Sigma(\mathscr{E})_{\mathrm{sing}}$.*
3. *There are $\tau_1, \ldots, \tau_r \in \Sigma$, such that*

$$X(\mathscr{E})_{\mathrm{sing}} = \bigcup_{\sigma \in \Sigma(\mathscr{E})_{\mathrm{sing}}} O(\sigma) = \bigcup_{i=1}^{r} V(\tau_i).$$

Moreover, $\min\{\dim(\sigma) : \sigma \in \Sigma(\mathscr{E})_{\mathrm{sing}}\} \geq 3$.

Proof. Let $\sigma \in \Sigma(\mathscr{E})_{\mathrm{free}}$. By Klyachko's compatibility condition (Proposition 2.3.12) there is a decomposition $\mathbb{E} = \bigoplus_{\alpha \in A} \mathbb{E}_\alpha$ with $A \subseteq M/M(\sigma)$ a multiset of size $\mathrm{rk}(\mathscr{E})$ such that for any $\rho \in \sigma(1)$,

$$E^\rho(i) = \bigoplus_{\alpha \in A,\, \langle \alpha, u_\rho \rangle \leq i} \mathbb{E}_\alpha.$$

Let τ be a face of σ and let $\mathrm{pr} : M/M(\sigma) \longrightarrow M/M(\tau)$ be the projection map. We denote by $B = \{\mathrm{pr}(\alpha) : \alpha \in A\}$ a multiset of size $\mathrm{rk}(\mathscr{E})$ and for any $\alpha \in A$, we set $\mathbb{E}'_{\mathrm{pr}(\alpha)} = \mathbb{E}_\alpha$. As for any $\rho \in \tau(1)$, we have $\langle \mathrm{pr}(\alpha), u_\rho \rangle = \langle \alpha, u_\rho \rangle$, we deduce that: for any $\rho \in \tau(1)$,

$$E^\rho(i) = \bigoplus_{\beta \in B,\, \langle \beta, u_\rho \rangle \leq i} \mathbb{E}'_\beta \,.$$

Thus $\tau \in \Sigma(\mathscr{E})_{\mathrm{free}}$, and then $\Sigma(\mathscr{E})_{\mathrm{free}}$ is a subfan of Σ.

Let $\tau \in \Sigma(\mathscr{E})_{\mathrm{sing}}$ and $\sigma \in \Sigma$ a cone containing τ. If $\sigma \in \Sigma(\mathscr{E})_{\mathrm{free}}$, by the first point we have $\tau \in \Sigma(\mathscr{E})_{\mathrm{free}}$. Thus, if $\tau \notin \Sigma(\mathscr{E})_{\mathrm{free}}$ then $\sigma \notin \Sigma(\mathscr{E})_{\mathrm{free}}$.

By [24, Corollary 5.5.20], we have $\dim X(\mathscr{E})_{\mathrm{sing}} \leq \dim X - 3$. As

$$X(\mathscr{E})_{\mathrm{sing}} = \bigcup_{\sigma \in \Sigma(\mathscr{E})_{\mathrm{sing}}} O(\sigma),$$

we deduce that for any $\sigma \in \Sigma(\mathscr{E})_{\mathrm{sing}}$, $\dim O(\sigma) \leq \dim X - 3$ and by the first point of the Orbit-Cone correspondence (Theorem 2.1.12), we get $\dim \sigma \geq 3$. □

5.1.2. Single orbit case. The aim of this section is to prove:

Proposition 5.1.3. *Let Σ be a smooth fan and let $\tau \in \Sigma$ with $\dim(\tau) \geq 3$. Then there is a torus equivariant reflexive sheaf $\mathscr{E}_\tau$ on X of rank $\dim(\tau) - 1$ such that $X(\mathscr{E}_\tau)_{\mathrm{sing}} = V(\tau)$.*

The construction is based on the following example introduced first by Hartshorne in [13, Example 1.9.1].

Example 5.1.4. We assume that $n \in \mathbb{N}$ satisfies $n \geq 3$. Let $(u_1, \ldots, u_n)$ be the standard basis of $\mathbb{Z}^n$ and $(e_1, \ldots, e_n)$ its dual basis. We denote by X the toric variety defined by the fan $\Sigma = \{\mathrm{Cone}(A) : A \subseteq \{u_1, \ldots, u_n\}\}$. The toric variety X is $\mathbb{C}^n$ and its torus is $(\mathbb{C}^*)^n$. For $i \in \{1, \ldots, n\}$, we set $\rho_i = \mathrm{Cone}(u_i)$ and $A_i = \{1, \ldots, n\} \setminus \{i\}$. Let $\mathscr{E}$ be the equivariant reflexive sheaf on X defined by the family of filtrations $(E, \{E^\rho(j)\})$ with

$$E^{\rho_i}(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ E_i & \text{if } j = -1 \\ E & \text{if } j \geq 0 \end{cases}$$

where $E = \mathrm{Span}(u_1, \ldots, u_{n-1})$, $E_i = \mathrm{Span}(u_i)$ for $i \in \{1, \ldots, n-1\}$ and $E_n = \mathrm{Span}(u_1 + \ldots + u_{n-1})$. For any $k \in \{1, \ldots, n\}$, we set $\sigma_k = \mathrm{Cone}(u_i : i \in A_k)$. On the cone σ_k, there is a T-eigenspace decomposition

$$E = \bigoplus_{i \in A_k} E_{[-e_i]}$$

with $E_{[-e_i]} = E_i$ such that, for any $l \in A_k$,

$$E^{\rho_l}(j) = \bigoplus_{i \in A_k,\, \langle -e_i, u_l \rangle \leq j} E_i \,.$$

Thus, $\Sigma(n-1) \subseteq \Sigma(\mathscr{E})_{\mathrm{free}}$. We assume that for $\sigma = \mathrm{Cone}(u_1, \ldots, u_n)$, there is an eigenspace decomposition $E = \bigoplus_{\alpha \in B} E_\alpha$ such that for any $i \in \{1, \ldots, n\}$,

$$E^{\rho_i}(j) = \bigoplus_{\alpha \in B,\, \langle \alpha, u_i \rangle \leq j} E_\alpha \,.$$

As $E^{\rho_i}(-1) = E_i$, we deduce that : for any $i \in \{1, \dots, n\}$, there is $\alpha \in B$ such that $E_\alpha = E_i$. So,

$$\dim\left(\bigoplus_{\alpha \in B} E_\alpha\right) \geq \dim\left(\bigoplus_{i=1}^{n} E_i\right) > \dim E;$$

this is a contradiction. Hence, by Proposition 2.3.12, $\mathrm{Cone}(u_1, \dots, u_n) \notin \Sigma(\mathscr{E})_{\mathrm{free}}$. This means that $\mathscr{E}$ is not locally free at the origin. ◊

Remark 5.1.5. If $\pi : \mathbb{C}^n \setminus \{0\} \longrightarrow \mathbb{P}^{n-1}$ denotes the quotient by homotheties, then the sheaf $\mathscr{E}$ of Example 5.1.4 is isomorphic to the extension to $\mathbb{C}^n$ by direct image of $\pi^* \mathscr{T}_{\mathbb{P}^{n-1}}$.

Proof of Proposition 5.1.3. Let $n = \dim(X)$ and $r = \dim(Y)$ with $Y = V(\tau)$. We assume that $\tau = \mathrm{Cone}(u_1, \dots, u_{n-r})$ and $\{\mathrm{Cone}(A) : A \subseteq \{u_1, \dots, u_n\}\} \subseteq \Sigma$ where $(u_1, \dots, u_n)$ is a $\mathbb{Z}$-basis of N. We set $\rho_i = \mathrm{Cone}(u_i)$ for $i \in \{1, \dots, n-r\}$. Let $E = \mathrm{Span}(u_1, \dots, u_{n-r-1})$, $E_i = \mathrm{Span}(u_i)$ for $i \in \{1, \dots, n-r-1\}$ and $E_{n-r} = \mathrm{Span}(u_1 + \ldots + u_{n-r-1})$. We denote by $\mathscr{E}$ the equivariant reflexive sheaf on X of rank $n-r-1$ given by the family of filtrations

$$E^{\rho_i}(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ E_i & \text{if } j = -1 \\ E & \text{if } j \geq 0 \end{cases}$$

for $i \in \{1, \dots, n-r\}$ and

$$E^{\rho}(j) = \begin{cases} 0 & \text{if } j < 0 \\ E & \text{if } j \geq 0 \end{cases}$$

for $\rho \in \Sigma(1) \setminus \tau(1)$. According to Example 5.1.4, $\tau \in \Sigma(\mathscr{E})_{\mathrm{sing}}$.

Let $\sigma = \mathrm{Cone}(u_1, \dots, u_n)$. The invariant affine open subset U_σ of X meets Y and we have compatible isomorphisms $U_\sigma \simeq \mathbb{C}^n$ and $U_\sigma \cap Y \simeq \mathbb{C}^r$ where $\mathbb{C}^r$ is identified with $\{0\} \times \mathbb{C}^r \subseteq \mathbb{C}^n$. We consider the $\mathbb{Z}$-linear map $\phi : N \longrightarrow \mathbb{Z}^{n-r-1}$ defined by

$$\phi(u_i) = \begin{cases} v_i & \text{if } 1 \leq i \leq n-r-1 \\ -(v_1 + \ldots + v_{n-r-1}) & \text{if } i = n-r \\ 0 & \text{if } n-r+1 \leq i \leq n \end{cases}$$

where $(v_1, \dots, v_{n-r-1})$ is a basis of $\mathbb{Z}^{n-r-1}$. We set $v_{n-r} = -(v_1 + \ldots + v_{n-r-1})$,

$$\Sigma_1 = \{\mathrm{Cone}(A) : A \subseteq \{u_1, \dots, u_n\} \text{ and } \{u_1, \dots, u_{n-r}\} \nsubseteq A\}$$

and

$$\Sigma_2 = \{\mathrm{Cone}(A) : A \subseteq \{v_1, \dots, v_{n-r}\}\}.$$

The map ϕ is compatible with the fan Σ_1 of $\mathbb{C}^n \setminus \mathbb{C}^r$ and the fan Σ_2 of $\mathbb{P}^{n-r-1}$. This induces a map

$$\pi : \mathbb{C}^n \setminus \mathbb{C}^r = (\mathbb{C}^{n-r} \setminus \{0\}) \times \mathbb{C}^r \longrightarrow \mathbb{C}^{n-r} \setminus \{0\} \longrightarrow \mathbb{P}^{n-r-1}$$

where the first map is the projection, and the second is the quotient by homotheties. Let $\mathscr{F}$ be the tangent sheaf of $\mathbb{P}^{n-r-1}$. Then $\pi^*\mathscr{F}$ is an equivariant locally free sheaf of $\mathbb{C}^n \setminus \mathbb{C}^r$ and its extension to $\mathbb{C}^n$ by direct image is an equivariant coherent and reflexive sheaf on $\mathbb{C}^n$ whose singular locus is $\mathbb{C}^r$. By using Example 2.3.11 and Proposition 4.1.1, the family of filtrations $(F, \{F^\rho(j)\}_{\rho \in \sigma(1), j \in \mathbb{Z}})$ of the sheaf $\pi^*\mathscr{F}$ on $U_\sigma \setminus Y$ is given by

$$F^{\rho_i}(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ \mathrm{Span}(v_i) & \text{if } j = -1 \\ \mathrm{Span}(v_1, \dots, v_{n-r-1}) & \text{if } j \geq 0 \end{cases}$$

if $i \in \{1, \ldots, n-r\}$ and by

$$F^\rho(j) = \begin{cases} 0 & \text{if } j < 0 \\ \mathrm{Span}(v_1, \ldots, v_{n-r-1}) & \text{if } j \geq 0 \end{cases}$$

if $\rho \in \sigma(1) \setminus \tau(1)$. Hence, $\pi^*\mathscr{F}$ is isomorphic to $\mathscr{E}_{|U_\sigma}$ on U_σ. Therefore, $\mathscr{E}_{|U_\sigma}$ has a singular locus equal to $U_\sigma \cap Y$. Thus,

$$\{\sigma' \in \Sigma : \tau \preceq \sigma' \preceq \sigma\} \subseteq \Sigma(\mathscr{E})_{\text{sing}}.$$

If $U_{\sigma'}$ is an invariant affine open subset of X which does not meet Y, then $\mathscr{E}_{|U_{\sigma'}}$ is isomorphic to the trivial sheaf of rank $n-r-1$. Therefore, $\mathscr{E}$ is locally free on $U_{\sigma'}$. Thus, $X(\mathscr{E})_{\text{sing}} = V(\tau)$. □

5.1.3. General case. Given a finite set of orbit closures of codimension at least 3, we show that:

Theorem 5.1.6. *Let X be a smooth toric variety with fan Σ. Let $\tau_1, \ldots, \tau_m \in \Sigma$ with $\dim(\tau_i) \geq 3$ such that for any $i, j \in \{1, \ldots, m\}$ with $i \neq j$, τ_i is not a proper face of τ_j. Then, there exists an equivariant reflexive sheaf $\mathscr{E}$ on X of rank $\sum_{i=1}^m \dim(\tau_i) - m$ such that*

$$X(\mathscr{E})_{\text{sing}} = \bigcup_{i=1}^m V(\tau_i).$$

For the proof, we use the following lemma.

Lemma 5.1.7. *Let $\mathscr{E}$ and $\mathscr{E}'$ be two equivariant reflexive sheaves on X such that $X(\mathscr{E})_{\text{sing}} = S_1$ and $X(\mathscr{E}')_{\text{sing}} = S_2$. We assume that S_1 and S_2 have no common irreducible component. Then, the sheaf $\mathscr{E} \oplus \mathscr{E}'$ of X satisfies $X(\mathscr{E} \oplus \mathscr{E}')_{\text{sing}} = S_1 \cup S_2$.*

Proof. For any $x \in X \setminus (S_1 \cup S_2)$, $(\mathscr{E} \oplus \mathscr{E}')_x$ is a free $\mathscr{O}_{X,x}$-module. So $X \setminus (S_1 \cup S_2) \subseteq X(\mathscr{E} \oplus \mathscr{E}')_{\text{free}}$ and $X(\mathscr{E} \oplus \mathscr{E}')_{\text{sing}} \subseteq S_1 \cup S_2$.

We now prove that $S_1 \cup S_2$ is included in $X(\mathscr{E} \oplus \mathscr{E}')_{\text{sing}}$. To do this, we use the following result of [24, Section 5.5] : $x \in X(\mathscr{E})_{\text{free}}$ if and only if $\mathrm{dh}(\mathscr{E}_x) = 0$ where dh is the homological dimension. Let $x \in S_1 \cup S_2 \setminus (S_1 \cap S_2)$. We can assume that $x \in S_1$, so $x \notin S_2$. As $(\mathscr{E} \oplus \mathscr{E}')_x = \mathscr{E}_x \oplus \mathscr{E}'_x$ and $\mathscr{E}'_x$ is free, any resolution of $\mathscr{E}_x \oplus \mathscr{E}'_x$ is of the form

$$E_j \oplus \mathscr{E}'_x \longrightarrow \ldots \longrightarrow E_0 \oplus \mathscr{E}'_x$$

for $E_j \longrightarrow \ldots \longrightarrow E_0$ a resolution of $\mathscr{E}_x$. Therefore, $\mathrm{dh}((\mathscr{E} \oplus \mathscr{E}')_x) \geq 1$. Hence, $S_1 \cup S_2 \setminus (S_1 \cap S_2) \subseteq X(\mathscr{E} \oplus \mathscr{E}')_{\text{sing}}$. By taking the Zariski closure of the inclusion $S_1 \cup S_2 \setminus (S_1 \cap S_2) \subseteq X(\mathscr{E} \oplus \mathscr{E}')_{\text{sing}}$ and using the fact that $X(\mathscr{E} \oplus \mathscr{E}')_{\text{sing}}$ is Zariski closed, we get $S_1 \cup S_2 \subseteq X(\mathscr{E} \oplus \mathscr{E}')_{\text{sing}}$. □

Proof of Theorem 5.1.6. We argue by induction on m. We assume that $m = 2$. According to Proposition 5.1.3, there are two sheaves $\mathscr{E}_1$ and $\mathscr{E}_2$ on X such that $X(\mathscr{E}_i)_{\text{sing}} = V(\tau_i)$ and $\mathrm{rk}(\mathscr{E}_i) = \dim(\tau_i) - 1$ for $i = 1, 2$. By Lemma 5.1.7, if we set $\mathscr{E} = \mathscr{E}_1 \oplus \mathscr{E}_2$, we get the result.

For $m \geq 3$, we assume that there is an equivariant reflexive sheaf $\mathscr{E}'$ on X of rank

$$\sum_{i=1}^{m-1} \dim(\tau_i) - (m-1)$$

such that

$$X(\mathscr{E}')_{\text{sing}} = S_1 = \bigcup_{i=1}^{m-1} V(\tau_i).$$

Let $\mathscr{E}''$ be an equivariant reflexive sheaf on X of rank $\dim(\tau_m) - 1$ such that $X(\mathscr{E}'')_{\text{sing}} = V(\tau_m)$. As S_1 and $V(\tau_m)$ have no common irreducible component, by Lemma 5.1.7, the sheaf $\mathscr{E} = \mathscr{E}' \oplus \mathscr{E}''$ satisfies $X(\mathscr{E})_{\text{sing}} = S_1 \cup V(\tau_m)$ and $\text{rk}(\mathscr{E}) = \sum_{i=1}^{m} \dim(\tau_i) - m$. This proves the theorem. $\square$

A

OUTLOOK

We give here a list of some problems that can be studied from the notions of Chapters 4 and 5 on pullbacks of sheaves and resolution of singularities.

A.1. Resolution of singularities

We use the assumptions of Proposition 4.2.4. Let $(u_1, \ldots, u_n)$ be a basis of N and $(e_1, \ldots, e_n)$ its dual basis such that $\{\text{Cone}(A) : A \subseteq \{u_1, \ldots, u_n\}\} \subseteq \Sigma$ and $\tau = \text{Cone}(u_1, \ldots, u_s)$ for $s \in \{2, \ldots, n\}$. We set $\rho_i = \text{Cone}(u_i)$ for $i \in \{0, 1, \ldots, n\}$ where $u_0 = u_1 + \ldots + u_s$. For any $m = m_1 e_1 + \ldots + m_n e_n \in M$,

$$E_m^{\rho_0} = \sum_{i_1+\ldots+i_s=0} E^{\rho_1}(m_1 + i_1) \cap \ldots \cap E^{\rho_s}(m_s + i_s) \, . \tag{A.1}$$

Example A.1.1. Applying (A.1) to Example 5.1.4 with $\tau = \text{Cone}(u_1, \ldots, u_n)$, we get

$$E^{\rho_0}(j) = \begin{cases} 0 & \text{if } j \leq -2 \\ E & \text{if } j \geq -1 \end{cases} \, .$$

Therefore, if $\sigma = \text{Cone}(u_0, u_1, \ldots, u_{n-1})$, there is a decomposition $E = \bigoplus_{i=1}^{n-1} E_{[-e_i]}$ with $E_{[-e_i]} = E_i$ such that for any $\rho \in \sigma(1)$,

$$E^{\rho}(j) = \bigoplus_{\substack{i=1 \\ \langle -e_i, u_\rho \rangle \leq j}}^{n-1} E_{[-e_i]} \, .$$

Hence, by Proposition 2.3.12, the sheaf $\mathscr{E}'$ is locally free on $X' = \text{Bl}_{V(\tau)}(X)$. ♢

In this example, we explain how blowing up the origin is enough to resolve the singularity of the sheaf given in Example 5.1.4. An application of Hironaka's resolution of indeterminacy locus gives:

Theorem A.1.2. *Let X be a smooth toric variety with fan Σ. Let $\mathscr{E}$ be an equivariant reflexive sheaf on X with singular locus $X(\mathscr{E})_{\text{sing}} = \bigcup_{i=1}^{r} V(\tau_i)$ for some $(\tau_i)_{1 \leq i \leq r} \in \Sigma^r$. Then, there is a sequence of at most p blow-ups along smooth irreducible torus invariant centers $\pi_i : X_i \longrightarrow X_{i-1}$ with $X_0 = X$ such that, if π denotes $\pi_p \pi_{p-1} \ldots \pi_1 : X_p \longrightarrow X$, the reflexive pullback $(\pi^* \mathscr{E})^{\vee\vee}$ is locally free on X_p.*

The number p given in this theorem is not explicit. So the natural question on the toric case would be to find an explicit bound on p according to the geometry of X and $\mathscr{E}$.

A.2. Pullbacks of sheaves along fibrations

A.2.1. Stability of sheaves in families. Let X be an n-dimensional smooth toric variety with torus T associated to a fan Σ. We recall that N is the lattice of one-parameter subgroups of T and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Let S be a scheme of finite type over $\mathbb{C}$, and let T_S be the relative torus of $\pi_S : X \times S \longrightarrow S$. An S-family of equivariant reflexive sheaves on X is a reflexive sheaf $\mathscr{E}$ on $X \times S$ with an action of the relative torus T_S compatible with the action on $X \times S$. Following [31, Proposition 3.13] and [25, Proposition 3.4], we have:

Proposition A.2.1. *The category of S-families of equivariant reflexive sheaves on X is equivalent to the category of reflexive sheaves $\mathscr{F}$ on S with collections of increasing filtrations*

$$\{\mathscr{F}^\rho_m : m \in M\}_{\rho \in \Sigma(1)}$$

indexed by the rays of Σ having the following properties:

1. *for all $m, m' \in M$ with $m \preceq_\rho m'$, there are injections $\mathscr{F}^\rho_m \longhookrightarrow \mathscr{F}^\rho_{m'}$ and $\mathscr{F}^\rho_m \longhookrightarrow \mathscr{F}$;*
2. *for each chain $\cdots \prec_\rho m_{i-1} \prec_\rho m_i \prec_\rho \cdots$ of elements of M, there exists $i_0 \in \mathbb{Z}$ such that $\mathscr{F}^\rho_{m_i} = 0$ for all $i \leq i_0$;*
3. *and, there are only finetely many $m \in M$ such that the morphism*

$$\bigoplus_{m' \prec_\rho m} \mathscr{F}^\rho_{m'} \longrightarrow \mathscr{F}^\rho_m$$

is not surjective.

Remark A.2.2. The conditions verified by these families of filtrations are similar to those given in Definition 2.3.5.

Proof. Let $\mathscr{E}$ be an S-family of equivariant reflexive sheaves on X. We denote by x_0 the identity element of T and $\mathscr{F}$ the reflexive sheaf $\mathscr{E}_{|x_0 \times S}$. For $\rho \in \Sigma(1)$, we set $\mathscr{F}^\rho = \Gamma(U_\rho \times S, \mathscr{E})$. As $\mathscr{E}$ is reflexive, the sheaf $\mathscr{E}$ on $X \times S$ is uniquely determined by $\mathscr{F}$ and the $\mathscr{F}^\rho$. By [32, Theorem 2.30], the action of T on $\mathscr{F}^\rho$ gives a decomposition into weight spaces

$$\mathscr{F}^\rho = \bigoplus_{m \in M} \mathscr{F}^\rho_m$$

where the $\mathscr{F}^\rho_m$ are $\mathscr{O}_S$-module of finite type. Hence, for any $m \in M$, $\mathscr{F}^\rho_m$ is a coherent sheaf over S. As $\mathscr{E}$ is reflexive, the sheaves $\mathscr{F}^\rho_m$ are reflexive.

The restriction of $\mathscr{E}$ to $x_0 \times S$ gives injections $\mathscr{F}^\rho_m \longhookrightarrow \mathscr{F}$ whose image depends only on the class $[m]$ in $M/(M \cap \rho^\perp)$. For $m' \in M \cap \rho^\vee$, multiplication by the character $\chi^{m'}$ gives a map from $\mathscr{F}^\rho_m$ to $\mathscr{F}^\rho_{m+m'}$. As this multiplication map is an isomorphism of $\mathscr{F}$ and the following diagram commutes,

$$\begin{array}{ccc} \mathscr{F}^\rho_m & \xrightarrow{\chi^{m'}} & \mathscr{F}^\rho_{m+m'} \\ \big\downarrow & & \big\downarrow \\ \mathscr{F} & \xrightarrow{\chi^{m'}} & \mathscr{F} \end{array}$$

we get an injection $\mathscr{F}^\rho_m \longhookrightarrow \mathscr{F}^\rho_{m+m'}$. This description makes it possible to establish the equivalence of categories. □

Let $\mathscr{E}$ be an S-family of equivariant reflexive sheaves on X. For any $s \in S$, we set

$$\mathscr{E}_s = \mathscr{E}_{|X \times \{s\}}.$$

Let $(\mathscr{F}, \{\mathscr{F}^\rho_m : m \in M\})$ be the collection of increasing filtrations associated to $\mathscr{E}$. Then, for any $s \in S$, the family of filtrations $(E_s, \{E^\rho_s(j)\})$ of the reflexive sheaf $\mathscr{E}_s$ is given by

$$E_s = \mathscr{F}(s) \quad \text{and} \quad E^\rho_s(j) = \mathscr{F}^\rho_m(s)$$

with $m \in M$ such that $\langle m, u_\rho\rangle = j$ where $\mathscr{F}(s)$ and $\mathscr{F}^\rho_m(s)$ are respectively the fiber of $\mathscr{F}$ and $\mathscr{F}^\rho_m$ at s defined in (2.13). We first observe that:

Lemma A.2.3. *Fix an ample divisor L on X. If for all $\rho \in \Sigma(1)$ and $j \in \mathbb{Z}$, the map $s \longmapsto \dim(E^\rho_s(j))$ is constant, then the set*

$$\{\mu_L((\mathscr{E}_s)_F) : s \in S,\ 0 \subsetneq F \subsetneq E_s\}$$

is finite.

Proof. The proof is similar to the proof of Lemma 2.3.19. For any $\rho \in \Sigma(1)$, there is $(j_\rho, J_\rho) \in \mathbb{Z}^2$ such that for any $s \in S$, $E^\rho_s(j) = \{0\}$ if $j < j_\rho$ and $E^\rho_s(j) = E_s$ if $j \geq J_\rho$. As

$$\{\dim(E^\rho_s(j) \cap F) - \dim(E^\rho_s(j-1) \cap F) : j \in \mathbb{Z}, s \in S, \text{ and } 0 \subsetneq F \subsetneq E_s\} \subseteq \{0, 1, \ldots, \mathrm{rk}(\mathscr{E})\},$$

we deduce that $\{\iota_\rho((\mathscr{E}_s)_F) : s \in S, 0 \subsetneq F \subseteq E_s\}$ is finite. Hence, the lemma follows from Formula (2.20). □

The previous lemma is the key to obtain a family version of Theorems 4.1.6 and 4.1.9. The *characteristic function* χ of an equivariant reflexive sheaf $\mathscr{G}$ with family of filtrations $(F, \{F^\rho(j)\})$ is the function

$$\begin{array}{rccc} \chi(\mathscr{G}): & M & \longrightarrow & \mathbb{Z}^{\sharp\Sigma(n)} \\ & m & \longmapsto & (\dim(F^\sigma_m))_{\sigma \in \Sigma(n)} \end{array}$$

where $F^\sigma_m = \bigcap_{\rho \in \sigma(1)} F^\rho(\langle m, u_\rho\rangle)$. The families that we will consider will satisfy one of the following:

(I) $\mathscr{E}$ is locally free on $X \times S$, or

(II) the characteristic function $(\chi(\mathscr{E}_s))_{s \in S}$ is constant.

Lemma A.2.4. *Let X be a smooth toric variety. Assume that $(\mathscr{E}_t)_{t \in S}$ satisfies (I) or (II). Then for all $\rho \in \Sigma(1)$ and $j \in \mathbb{Z}$, $s \longmapsto \dim(E^\rho_s(j))$ is constant.*

Proof. In the case that the family satisfies (I), by [31, Proposition 3.13] (Klyachko's compatibility condition for S-families of locally free sheaves), for any $\sigma \in \Sigma(n)$, there is a multiset $A_\sigma \subseteq M$ of size $\mathrm{rk}(\mathscr{E})$ such that for any $m \in M$, $\mathscr{F}^\rho_m$ is a locally free sheaf of rank

$$|\{\alpha \in A_\sigma : \langle \alpha, u_\rho\rangle \leq \langle m, u_\rho\rangle\}|\,.$$

As for any $s \in S$ and $m \in M$, $\dim(\mathscr{F}^\rho_m(s)) = \mathrm{rk}(\mathscr{F}^\rho_m)$, we deduce that the map

$$s \longmapsto \dim(E^\rho_s(\langle m, u_\rho\rangle))$$

is constant.

We now assume (II). For any $\sigma \in \Sigma(n)$, the set $\{u_\rho : \rho \in \sigma(1)\}$ is a basis of N. Then, for any $\rho \in \sigma(1)$ and any $j \in \mathbb{Z}$, we can find an element $m \in M$ such that for all $s \in S$,

$$\langle m, u_\rho\rangle = j$$

and for $\rho' \in \sigma(1) \setminus \{\rho\}$,

$$E^{\rho'}_s(\langle m, u_{\rho'}\rangle) = E_s.$$

This can be made uniform in s as follows: by (II), we can fix $m' \in M$ such that $E^{\sigma}_{s,m'} = E_s$ for all $s \in S$. This implies that for $\rho' \in \sigma(1)$, $E^{\rho'}_s(\langle m', u_{\rho'}\rangle) = E_s$. Then, define

$$m = ju^*_{\rho} + \sum_{\rho' \neq \rho} \langle m', u_{\rho'}\rangle u^*_{\rho'}$$

where $\{u^*_{\rho'} : \rho' \in \sigma(1)\}$ is the dual basis of $\{u_{\rho'} : \rho' \in \sigma(1)\}$. But then

$$E^{\sigma}_{s,m} = \bigcap_{\rho' \in \sigma(1)} E^{\rho'}_s(\langle m, u_{\rho'}\rangle) = E^{\rho}_s(j)$$

and (II) implies the result. □

Let $\pi : X' \longrightarrow X$ be a toric fibration between two smooth toric varieties and $(\mathscr{E}_t)_{t \in S}$ be a family of reflexive sheaves on X satisfying (I) or (II). Assume that for all $t \in S$, $\mathscr{E}_t$ is stable on (X, L). From Lemma A.2.4 and Lemma A.2.3, we deduce that the ε_0 in the proof of Theorem 4.1.6 can be taken uniformly in $t \in S$. Note for this that in the expansions in ε of formula (2.20) for the slopes $\mu_{L_\varepsilon}(\mathscr{E}'_t)$, the terms $\iota_\rho(\mathscr{E}'_t)$ do not vary with ε, only the terms $\deg_{L_\varepsilon}(D_\rho)$ do. Similarly, we can take ε_0 uniform in Theorem 4.1.9 if all $\mathscr{E}_t$ are assumed to be sufficiently smooth on (X, L).

We deduce from this the existence of injective maps between components of the moduli spaces of stable equivariant reflexive sheaves on (X, L) and on (X', L_ε), for ε small enough. One can consider the moduli space of equivariant stable reflexive sheaves on (X, L) with fixed characteristic function χ introduced in [25], denoted $\mathscr{N}^{\mu s}_{\chi}(X, L)$. As χ determines the Chern character ([23] and [25, Section 3.4]), and thus the Hilbert polynomial by Hirzebruch-Riemann-Roch, we deduce that the reflexive pullback induces an injective map for $\varepsilon \ll 1$:

$$\pi^* : \mathscr{N}^{\mu s}_{\chi}(X, L) \longrightarrow \mathscr{N}^{\mu s}_{P'}(X', L_\varepsilon)$$

where P' denotes the Hilbert polynomial with respect to L_ε of any element $(\pi^*\mathscr{E})^{\vee\vee}$ with characteristic function χ. In fact, if we denote P_χ the Hilbert polynomial induced by χ, we expect that this map is actually defined on

$$\mathscr{N}^{\mu s}_{P}(X, L) = \bigcup_{P_\chi = P} \mathscr{N}^{\mu s}_{\chi}(X, L)$$

the moduli of stable equivariant reflexive sheaves with Hilbert polynomial P. In the same way, fixing the total Chern class, one should obtain maps between the moduli spaces of equivariant and stable locally free sheaves. Those spaces should be obtained as open sub-schemes of the moduli spaces constructed in [31]. We believe that those maps deserve further study and will come back to them in future research.

A.2.2. Pullback of sheaves. Theorem 4.1.9 and 4.2.10 are given in the toric setting. Our goal is to see if these theorems remain true without the toric assumptions on sheaves or in the case of normal projective varieties. In the toric setting, it is Lemma 2.3.19 which simplifies the study. In other cases, the idea will be to study the stability of $\mathscr{E}'$ by using

$$\sup\{\mu_L(\mathscr{F}) : \mathscr{F} \subseteq \mathscr{E} \text{ a subsheaf with } 0 < \operatorname{rk}(\mathscr{F}) < \operatorname{rk}(\mathscr{E})\}$$

and the length of the graded object $\operatorname{Gr}_L(\mathscr{E})$ of $\mathscr{E}$ with respect to L.

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COLLEGE DOCTORAL BRETAGNE — MATHSTIC BRETAGNE OCEANE

UBO Université de Bretagne Occidentale

Titre : Faisceaux équivariants stables sur les variétés toriques

Mots-clés : Variétés toriques, faisceaux réflexifs équivariants, stabilité au sens de la pente

Résumé : Le problème abordé dans cette thèse est celui de la construction de faisceaux réflexifs équivariants stables sur les variétés toriques. Ce travail est motivé par la question de classification des fibrés vectoriels sur les variétés complexes compactes. Cette thèse est formée de trois parties. Dans la première partie, nous étudions la stabilité des faisceaux tangents logarithmiques équivariants $\mathscr{T}_X(-\log D)$ où X est une variété torique projective et D un diviseur réduit. Le résultat principal de cette partie est la classification complète des diviseurs réduits D et des polarisations L sur X tels que le faisceau tangent logarithmique $\mathscr{T}_X(-\log D)$ est (semi)stable par rapport à L lorsque X est de rang de Picard deux. Dans la deuxième partie, pour un faisceau réflexif équivariant sur une variété torique polarisée, nous étudions la stabilité de l'enveloppe réflexive de son pullback le long d'une fibration torique. On montre que la stabilité (resp. l'instabilité) est préservée par certaines polarisations dites adiabatiques. Dans le cas où le faisceau est localement libre et semistable, nous donnons une condition nécessaire et suffisante sur son objet gradué pour que son pullback réflexif devienne stable. Dans la dernière partie, nous étudions le lieu singulier des faisceaux réflexifs équivariants. Nous construisons explicitement un faisceau réflexif ayant pour lieu singulier une sous-variété irréductible de codimension au moins trois.

Title: Stable equivariant sheaves on toric varieties

Keywords: Toric varieties, equivariant reflexive sheaves, slope-stability

Abstract: This PhD thesis deals with the problem of construction of stable equivariant reflexive sheaves on toric varieties. This work is motivated by the question of classification of vector bundles on compact complex manifolds. This thesis consists of three parts. In the first part, we study the stability of equivariant logarithmic tangent sheaves $\mathscr{T}_X(-\log D)$ where X is a projective toric variety and D a reduced divisor. The main result of this part is the classification of reduced divisors D and polarizations L on X such that the equivariant logarithmic tangent sheaf $\mathscr{T}_X(-\log D)$ is (semi)stable with respect to L when X is a smooth toric variety of Picard rank two. In the second part, for an equivariant reflexive sheaf on a polarized toric variety, we study the stability of its reflexive pullback along a toric fibration. We show that stability (resp. unstability) is preserved under such pullbacks by some adiabatic polarizations. In the case where the sheaf is semistable, under local freeness assumptions, we provide a necessary and sufficient condition on the graded object to ensure stability of the pulled back sheaf. In the last part, we study the singular locus of equivariant reflexive sheaves. We construct an explicit equivariant reflexive sheaf whose singular locus is an irreducible subvariety of codimension at least three.